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Quantum Electrodynamics at Large Distances I: Extracting the Correspondence-Principle Part. *

Takahiro Kawai

*Research Institute for Mathematical Sciences**Kyoto University**Kyoto 606-01 JAPAN*

Henry P. Stapp

*Lawrence Berkeley Laboratory**University of California**Berkeley, California 94720*

Abstract

The correspondence principle is important in quantum theory on both the fundamental and practical levels: it is needed to connect theory to experiment, and for calculations in the technologically important domain lying between the atomic and classical regimes. Moreover, a correspondence-principle part of the S-matrix is normally separated out in quantum electrodynamics in order to obtain a remainder that can be treated perturbatively. But this separation, as usually performed, causes an apparent breakdown of the correspondence principle and the associated pole-factorization property. This breakdown is spurious. It is shown in this article, and a companion, in the context of a special case, how to extract a distinguished part of the S-matrix that meets the correspondence-principle and pole-factorization requirements. In a second companion paper the terms of the remainder are shown to vanish in the appropriate macroscopic limits. Thus this work validates the correspondence principle and pole factorization in quantum electrodynamics, in the special case treated here, and creates a needed computational technique.

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1. Introduction

The correspondence principle asserts that the predictions of quantum theory become the same as the predictions of classical mechanics in certain macroscopic limits. This principle is needed to explain why classical mechanics works in the macroscopic domain. It also provides the logical basis for using the language and concepts of classical physics to describe the experimental arrangements used to study quantum-mechanical effects.

It is primarily within quantum electrodynamics that the correspondence principle must be verified. For it is quantum electrodynamics that controls the properties of the measuring devices used in these experimental studies.

In quantum electrodynamics the correspondence-principle has two aspects. The first pertains to the electromagnetic fields generated by the macroscopic motions of particles: these fields should correspond to the fields generated under similar conditions within the framework of classical electrodynamics. The second aspect pertains to motion of the charged particles: on the macroscopic scale these motions should be similar to the motions of charged particles in classical electromagnetic theory.

The pole-factorization property is the analog in quantum theory of the classical concept of the stable physical particle. This property has been confirmed in a variety of rigorous contexts^{1,2,3} for theories in which the vacuum is the only state of zero mass. But calculations^{4,5,6} have indicated that the property fails in quantum electrodynamics, due to complications associated with infrared divergences. Specifically, the singularity associated with the propagation of a physical electron has been computed to be not a pole. Yet if the mass of the physical electron were m and the dominant singularity of a scattering function at $p^2 = m^2$ were not a pole then physical electrons would, according to theory, not propagate over laboratory distances like stable particles, contrary to the empirical evidence.

This apparent difficulty with quantum electrodynamics has been extensively studied^{7,8,9}, but not fully clarified. It is shown here, at least in the context of a special case that is treated in detail, that the apparent failure in quantum electrodynamics of the classical-type spacetime behaviour of electrons and positrons in the macroscopic regime is due to approximations introduced to

cope with infrared divergences. Those divergences are treated by factoring out a correspondence-principle part, before treating the remaining part perturbatively. It will be shown here, at least within the context of the case examined in detail, that if an accurate correspondence-principle part of the photonic field is factored out then the required correspondence-principle and pole-factorization properties do hold. The apparent failure of these latter two properties in the cited references are artifacts of approximations that are not justified in the context of the calculation of macroscopic spacetime properties: some factors $\exp ikx$ are replaced by substitutes that introduce large errors for small k but very large x .

The pole-factorization theorem, restricted to the simplest massive-particle case, asserts the following: Suppose the momentum-space scattering function for a process $(1 + 2 \leftarrow 3 + x)$ has a nonzero connected component

$$S_c(p_1, p_2; p_3, p_x) \delta^4(p_1 + p_2 - p_3 - p_x),$$

and that the scattering function for a process $(x + 4 \leftarrow 5 + 6)$ has a nonzero connected component

$$S_c(p_x, p_4; p_5, p_6) \times \delta^4(p_x + p_4 - p_5 - p_6).$$

Then, according to the theorem, the three-to-three scattering function

$$S_c(p_1, p_2, p_4; p_3, p_4, p_5) \times \delta^4(p_1 + p_2 + p_4 - p_3 - p_5 - p_6)$$

must have the form

$$\frac{N(p_1, p_2, p_4; p_3, p_4, p_5)}{(p_1 + p_2 - p_3)^2 - m^2 + i0},$$

where m is the mass of particle x , and the residue of the pole is a (known) constant times

$$S_c(p_1, p_2; p_3, p_x) S_c(p_x, p_4; p_5, p_6) \times \delta(p_1 + p_2 + p_4 - p_3 - p_5 - p_6),$$

where $p_x = p_1 + p_2 - p_3 = p_5 + p_6 - p_4$.

The physical significance of this result arises as follows. Suppose we form wave packets $\psi_i(p_i)$ ($i = 1, \dots, 6$) for the six (external) particles of the three-to-three process. Let these momentum-space wave packets be nonzero at a set of values \bar{p}_i such that

$$\bar{p}_1 + \bar{p}_2 - \bar{p}_3 = \bar{p}_x = \bar{p}_5 + \bar{p}_6 - \bar{p}_4$$

where $(\bar{p}_x)^2 = m^2$. Suppose the corresponding free-particle coordinate-space wave packets $\tilde{\psi}_i(x_i, t)$ for these six particles are all large at the origin of spacetime. Now translate the wave packets of particles 1, 2, and 3 by the spacetime distance $\lambda\bar{p}$, and let λ tend to infinity. Then $\lambda^{3/2}$ times this $3 \rightarrow 3$ transition amplitude must, according to the theorem, tend to a limit that is a (known) constant times the product of the two scattering amplitudes,

$$A_D = \int d^3p_1 \int d^3p_2 \int d^3p_3 \psi_1^*(p_1) \psi_2^*(p_2) \psi_3(p_3) S_c(p_1, p_2; p_3, \bar{p}_x)$$

and

$$A_P = \int d^3p_4 d^3p_5 d^3p_6 \psi_4^*(p_4) \psi_5(p_5) \psi_6(p_6) S_c(\bar{p}_x, p_4; p_5, p_6).$$

This result has the following physical interpretation: the transition amplitude A_P is the amplitude for *producing* a particle x of momentum \bar{p}_x , and the amplitude A_D is the amplitude for *detecting* this particle. The fall-off factor $\lambda^{-3/2}$ becomes λ^{-3} when one passes from amplitudes to probabilities, and this factor λ^{-3} is what would be expected on purely geometric grounds in classical physics, if the intermediate particle x produced by the production process, and detected by the detection process travelled, in the asymptotic regime, on a straight line in spacetime with four-velocity $v_x = p_x/m \simeq \bar{v}_x = \bar{p}_x/m$.

This λ^{-3} fall-off property is also what is observed empirically for both neutral and charged particles travelling over large distances in free space. On the other hand, computations^{4,5,6} in QED have shown that if quasi-classical parts are factored off in the usual momentum-space manner then in the remainder the singularities associated with the propagation of charged particles have, instead of the pole form $(p^2 - m^2 + i0)^{-1}$, rather a form $(p^2 - m^2 + i0)^{-1-\beta}$, where β is nonzero and of order $\alpha \approx 1/137$. Such a form would entail that electrons and positrons would not behave like stable particles: they would evoke weaker and weaker detection signals (or seems to disappear) for $\beta < 0$, or evoke stronger and stronger detection signals for $\beta > 0$, as their distance from the source increases.

Such effects are not observed empirically. Hence β must be zero (or at least close to zero), in apparent contradiction to the results of the cited QED calculations.

For the idealized case in which all particles have nonzero mass the pole-factorization theorem has been proved in many ways. The simplest “proof” is

simply to add up all of the Feynman-graph contributions that have the relevant pole propagator $(p^2 - m^2 + i0)^{-1}$, and observe that the residue has the required form. Proofs not relying on perturbation theory have been given in the frameworks of quantum field theory¹, constructive field theory², and S -matrix theory³.

In quantum electrodynamics if the particle x is *charged* then at least one other charged particle must either enter or leave each of the two subprocess, in order for charge to be conserved. If there is a deflection of this charged particle in either of these two subprocesses then bremsstrahlung radiation will be emitted by that process. As $\lambda \rightarrow \infty$ the number of photons radiated will tend to infinity. Thus in place of the two simple $2 \rightarrow 2$ sub-processes considered in the example discussed above one must include in QED the bremsstrahlung photons radiated at each of the two subprocesses.

Bremsstrahlung photons were in fact taken into account in the earlier cited works^{1,2,3}. However, in those works it was assumed, in effect, that all of these photons were emitted from a neighborhood of the origin in spacetime. This imprecision in the positioning of the sources of the bremsstrahlung radiation arose from the use of a basically momentum-space approach.

It is clear that coordinate-space should provide a more suitable framework for accurately positioning the sources of the radiated photons. Indeed, it turns out that it is sufficient to place the sources of the (real and virtual) bremsstrahlung photons at the physically correct positions in coordinate space in order to establish the validity of a pole-factorization property in QED, at least in the special case that we study in detail in this paper.

Examination of the work of Kibble⁴ shows that there is, in the case he treated, also another problem. In that case some of the charged-particle lines extend to plus or minus infinity. At one point in the calculation, a factor $(p_\mu/p \cdot k)(e^{ipx_1} - e^{ikx_2})$ initially associated with such a line, where x_1 and x_2 represent the two ends of the charged particle line, is replaced by a single one of the two terms: the other term, corresponding to the point $x_i = \infty$, is simply dropped. Yet dropping this term alters the character of the behavior at $k = 0$: the original product of this form with k^μ tends to zero as k vanishes, but to plus or minus unity if a term is dropped.

It turns out that this treatment of the contributions corresponding to points at infinity leads to serious ambiguities.⁷ To avoid such problems, and keep everything finite and well defined in the neighborhood of $k = 0$, we shall consider the case of a “confined charge”; i.e., a case in which a charge travels around a closed loop in spacetime, in the Feynman sense: a backward moving electron is interpreted as a forward moving positron. In particular, we shall consider an initial graph in which the charge travels around a closed triangular loop $L(x_1, x_2, x_3)$ that has vertices at spacetime points x_1 , x_2 , and x_3 . These three vertices represent points where “hard” photons interact. (Actually, each x_i will correspond to a *pair* of hard-photon vertices, but we shall, in this introduction, ignore this slight complication, and imagine the two hard photons to be attached to the same vertex of the triangle.) We must then consider the effects of inserting arbitrary numbers of “soft photon” vertices into this hard-photon triangle in all possible ways. The three hard-photon vertices x_i are held fixed during most of the calculation. At the end one must, of course, multiply this three-point coordinate-space scattering function by the coordinate-space wavefunctions of the external particles connected at these three spacetime points, and then integrate over all possible values of x_1 , x_2 , and x_3 .

As in the two-vertex example given above, we are interested in the behavior in the limit in which (x_1, x_2, x_3) is replaced by $(\lambda x_1, \lambda x_2, \lambda x_3)$ and λ tends to infinity. The physically expected fall-off rate is now $(\lambda^{-3/2})^3$, with one geometric fall-off factor $\lambda^{-3/2}$ for each of the three intermediate charged-particle lines.

This $\lambda^{-9/2}$ fall off is exactly the coordinate-space fall-off that arises from a Feynman function corresponding to graph consisting of external lines connected to the three vertices of a triangle of internal lines. The singularity in momentum space corresponding to such a simple triangle graph is $\log \varphi$, where

$$\varphi = \varphi(q_1, q_2, q_3) = 0$$

is the so-called Landau-Nakanishi (or, for short, Landau) triangle-diagram singularity surface. Here the q_i are the momenta entering the three vertices, and they are subject to the momentum-energy conservation law $q_1 + q_2 + q_3 = 0$.

In close analogy to the single-pole case discussed earlier, the discontinuity of the full scattering function across this $\log \varphi$ surface at $\varphi = 0$ is, in theories with no massless particles, a (known) constant times a product of three scattering

functions, one corresponding to each of the three vertices of the triangle:

$$\text{disc}S|_{\varphi=0} = \text{const.} \times S_1 S_2 S_3.$$

It will be shown in these papers that this formula for the discontinuity around the triangle-diagram singularity surface $\varphi = 0$ holds also in quantum electrodynamics to every order of the perturbative expansion in the nonclassical part of the photon field. The situation is more complicated than in the massive-particle case because now an infinite number of singularities of different types all coincide with $\varphi = 0$. It will be shown that many of these do not contribute to the discontinuity at $\varphi = 0$, because the associated discontinuities contain at least one full power of φ , and that all of the remaining contributions are parts of the discontinuity function given above.

Another complication is that an infinite number of photons are radiated from each of the three vertices of the triangle. In our treatment these photons are contained in the (well-defined) classical part of the photon field. The contributions from these photons depend on the locations of the vertices x_i , and are incorporated *after* the transformation to coordinate space.

This focus on the triangle-graph process means that we are dealing here specifically with the charge-zero sector. But the scattering functions for charged sectors can be recovered by exploiting the proved pole-factorization property. It is worth emphasizing, in this connection, that a straight-forward application of perturbation theory in the triangle-graph case does *not* yield the pole-factorization property, even though the triangle graph represents a process in the charge-zero sector. Just as in the charged sectors, it is still necessary to separate out the part corresponding to the classical photons. If one does not, then the first-order perturbative term gives a singularity of the form¹⁰ $(\log\varphi)^2$, instead of the physically required form $\log\varphi$. It is worth emphasizing that we do not neglect “small” terms in denominators, but keep everything exact. Indeed, it is important that we do so, because these small terms are essential to the validity of law of conservation of charge, which we use extensively.

In the foregoing discussion we have focussed on the pole-factorization-theorem aspects of our work. But the paper contains much more. It provides the mathematical machinery needed to apply quantum electrodynamics in the mesoscopic and macroscopic regimes where charged particles move be-

tween interaction regions that are separated by distances large enough for the long-distance particle-type behaviours of these particles to begin to manifest themselves. That is, this paper establishes a formalism that allows quantum electrodynamics to be accurately applied to the transitional domain lying between the quantum and classical regimes. The machinery displays in a particularly simply and computationally useful form the infrared-dominant “classical” part of the electromagnetic field, while maintaining good mathematical control over the remaining “quantum” part.

This work is based on the separation defined in reference 11 of the electromagnetic interaction operator into its “classical” and “quantum” parts. This separation is made in the following way. Suppose we first make a conventional energy-momentum-space separation of the (real and virtual photons) into “hard” and “soft” photons, with hard and soft photons connected at “hard” and “soft” vertices, respectively. The soft photons can have small energies and momenta on the scale of the electron mass, but we shall not drop any “small” terms. Suppose a charged-particle line runs from a hard vertex x^- to a hard vertex x^+ . Let soft photon j be coupled into this line at point x_j , and let the coordinate variable x_j be converted by Fourier transformation to the associated momentum variable k_j . Then the interaction operator $-ie\gamma_{\mu_j}$ is separated into its “classical” and “quantum” parts by means of the formula

$$-ie\gamma_{\mu_j} = C_{\mu_j} + Q_{\mu_j}, \quad (1.1)$$

where

$$C_{\mu_j} = -ie \frac{z_{\mu_j}}{z \cdot k_j} \not{k}_j, \quad (1.2)$$

and $z = x^+ - x^-$.

This separation of the interaction allows a corresponding separation of soft photons into “classical” and “quantum” photons: a “quantum” photon has a quantum coupling on at least one end; all other photons are called “classical” photons. The full contribution from all classical photons is represented in an extremely neat and useful way. Specialized to our case of a single charged-particle loop $L(x_1, x_2, x_3)$ the key formula reads

$$F_{op}(L(x_1, x_2, x_3)) =: U(L(x_1, x_2, x_3)) F'_{op}(L(x_1, x_2, x_3)) :. \quad (1.3)$$

Here $F_{op}(L(x_1, x_2, x_3))$ is the Feynman *operator* corresponding to the sum of contributions from *all* photons coupled into the charged-particle loop $L(x_1, x_2, x_3)$, and $F'_{op}(L(x_1, x_2, x_3))$ is the analogous operator if all contributions from classical photons are excluded. The operators F_{op} and F'_{op} are both normal ordered operators: i.e., they are operators in the asymptotic-photon Hilbert space, and the destruction operators of the incoming photons stand to the right of the creation operators of outgoing photons. On the right-hand side of (1.3) all of the contributions corresponding to classical photons are included in the unitary-operator factor $U(L)$ defined as follows:

$$U(L) = e^{<a^* \cdot J(L)>} e^{-\frac{1}{2} <J^*(L) \cdot J(L)>} e^{-<J^*(L) \cdot a>} e^{i\Phi(L)}. \quad (1.4)$$

Here, for any a and b , the symbol $<a \cdot b>$ is an abbreviation for the integral

$$<a \cdot b> \equiv \int \frac{d^4 k}{(2\pi)^4} 2\pi \theta(k_0) \delta(k^2) a_\mu(k) (-g^{\mu\nu}) b_\nu(k), \quad (1.5)$$

and $J(L, k)$ is formed by integrating $\exp ikx$ around the loop L :

$$J_\mu(L, k) \equiv \int_L dx_\mu e^{ikx}. \quad (1.6)$$

This classical current $J_\mu(L)$ is conserved:

$$k^\mu J_\mu(L, k) = 0. \quad (1.7)$$

The a^* and a in (1.4) are photon creation and destruction operators, respectively, and $\Phi(L)$ is the classical action associated with the motion of a charged classical particle along the loop L :

$$\Phi(L) = \frac{(-ie)^2}{8\pi} \int_L dx'_\mu g^{\mu\nu} \int_L dx''_\nu \delta((x' - x'')^2) \quad (1.8)$$

The operator $U(L)$ is *pseudo* unitary if it is written in explicitly covariant form, but it can be reduced to a strictly unitary operator using by (1.7) to eliminate all but the two transverse components of $a_\mu(k)$, $a_\mu^*(k)$, $J_\mu(k)$, and $J_\mu^*(k)$.

The colons in (1.3) indicate that the creation-operator parts of the normal-ordered operator F'_{op} are to be placed on the left of $U(L)$.

The unitary operator $U(L)$ has the following property:

$$U(L)|vac> = |C(L)>. \quad (1.9)$$

Here $|vac\rangle$ is the photon vacuum, and $|C(L)\rangle$ represents the normalized coherent state corresponding to the classical electromagnetic field radiated by a charged classical point particle moving along the closed spacetime loop L , in the Feynman sense.

The simplicity of (1.3) is worth emphasizing: it says that the complete effect of all classical photons is contained in a simple multiplicative factor that is independent of the quantum-photon contributions: this factor is a well-defined unitary operator that depends only on the (three) hard vertices x_1, x_2 , and x_3 . It is independent of the remaining details of $F'_{op}(L(x_1, x_2, c_3))$, even though the classical couplings are originally interspersed in all possible ways among the quantum couplings that appear in $F'_{op}(L(x_1, x_2, x_3))$. The operator $U(L)$ supplies the classical bremsstrahlung-radiation photons associated with the deflections of the charged particles that occur at the three vertices, x_1, x_2 , and x_3 .

Block and Nordsieck¹² have already emphasized that the infrared divergences arise from the classical aspects of the electromagnetic field. This classical component is exactly supplied by the factor $U(L)$. One may therefore expect the remainder $F'_{op}(L(x_1, x_2, x_3))$ to be free of infrared problems: if we transform $F'_{op}(L(x_1, x_2, x_3))$ into momentum space, then it should satisfy the usual pole-factorization property. A primary goal of this work is to show that this pole-factorization property indeed holds. To recover the physics one transforms F'_{op} to coordinate space, and then incorporates the real and virtual classical photons by using 1.3 and 1.4.

The plan of the paper is as follows. In the following section 2 rules are established for writing down the functions of interest directly in momentum space. These rules are expressed in terms of operators that act on momentum-space Feynman functions and yield momentum-space functions, with classical or quantum interactions inserted into the charged-particle lines in any specified desired order.

It is advantageous always to sum together the contributions corresponding to all ways in which a photon can couple with C-type coupling into each individual side of the triangle graph G . This sum can be expressed as a sum of just two terms. In one term the photon is coupled at one endpoint, x^+ , of this side

of G , and in the other term the photon is coupled into the other end point, x^- , of this side of G . Thus all C-type couplings become converted into couplings at the hard-photon vertices of the original graph G .

This conversion introduces an important property. The charge-conservation (or gauge) condition $k^\mu J_\mu = 0$ normally does not hold in quantum electrodynamics for individual graphs: one must sum over all ways in which the photon can be inserted into the graph. But in the form we use, with each quantum vertex Q coupled into the interior of a line of G , but each classical vertex C placed at a hard-photon vertex of G , the charge-conservation equation (gauge invariance) holds for each vertex separately: $k^\mu J_\mu = 0$ for each vertex.

In section 3 the modification of the charged-particle propagator caused by inserting a single quantum vertex Q_μ into a charged-particle line is studied in detail. The resulting (double) propagator is re-expressed as a sum of three terms. The first two are “meromorphic” terms having poles at $p^2 = m^2$ and $p^2 = m^2 - 2pk - k^2$, respectively, in the variable p^2 . Because of the special form of the quantum coupling Q_μ each residue is of first order in k , relative to what would have been obtained with the usual coupling γ_μ . This extra power of k will lead to the infrared convergence of the residues of the pole singularities.

The third term is a nonmeromorphic contribution. It is a difference of two logarithms. This *difference* has a power of k that renders the contribution infrared finite.

In section 4 the results just described are used to study the function corresponding to a graph g that is formed by inserting into the triangle graph G a single quantum photon that has Q-type interactions at each end. In order to treat in a rigorous way the contribution from the neighborhood of the point $k = 0$ we introduce polar coordinates $k = r\Omega$, $\Omega\tilde{\Omega} \equiv \Omega_0^2 + \vec{\Omega} \cdot \vec{\Omega} = 1$. For the meromorphic contributions it is found that the integrand of the integral that defines the residue behaves like rdr near the end point $r = 0$, and that the compact domain of integration in the variable Ω can be distorted away from all singularities. This shows that there is no infrared divergence. The two meromorphic contributions from each end of the photon line lead to four contributions to $F(D')$. One of them gives the normal $\log \varphi$ singularity on the Landau triangle-diagram surface $\varphi = 0$, and the other three give weaker singularities. The contributions from

the nonmeromorphic contributions also give weaker singularities.

The aim of the remaining sections is basically to prove that the analogous results hold for all graphs g constructed from the original triangle graph G by the addition of any number of quantum-photon lines. In the process of proving this, we construct the foundation of an efficient general machinery for computing, in quantum electrodynamics, the physical-region singularity structure, or, equivalently, an accurate representation of the large-distance spacetime behavior.

In section 5 we examine the *generalized propagator* that corresponds to charged-particle propagation between two hard-photon vertices x and y with an arbitrary number of Q-type insertions. The meromorphic part is exhibited explicitly: there is one pole term for each of the original energy denominators. The residues factorize, and each of the two factors (unless it is unity) has *one* factor of k_i beyond what would occur if the couplings were the original γ_μ couplings. This single extra factor of k_i in each residue factor will lead to infrared convergence of the meromorphic parts.

This infrared convergence result, for any graph g obtained by inserting a set of internal quantum photons into the triangle graph G , is proved in sections 6 and 7, subject to the assumption that, in analogy to what occurred in the simple case treated in section 4, the Ω contours can be distorted so as to avoid all singularities of the residue factors. This distortion assumption reduces the problem to that of counting powers of r . However, it is not sufficient merely to count overall powers of r . One must show that, for every possible way in which the variables k_i can tend to zero, there is convergence of every sub-integral. Our proof that this convergence property holds can be regarded as a systematization and confirmation of the argument for infrared convergence given by Grammer and Yennie¹³. The problem is non-trivial because for every $n > 0$ there are terms with n factors of the form $d^4 k_i / D_i$, where the denominator D_i is of fourth order in the k_i , for small k_i , but there are at most six extra numerator factors k_i : the problem is to show that these few factors k_i are enough to block all of the $4n$ potential logarithmic divergences. The problem is basically a topological one.

Section 8 gives a description of a result that asserts that the contours in

Ω_i -space can be distorted away from all singularities of the residue factors and photon propagators. The proof of this result is given in a companion paper¹⁴.

In section 9 the results of the earlier sections are gathered together and extended to give the result that singularities on the triangle-diagram surface coming from the meromorphic parts of the contributions arising from the quantum photons are no stronger than $\log\varphi$. References are made to a second companion paper¹⁵, which proves pertinent properties of some integrals that occur in this work. In sections 10 and 11 the similar results for the nonmeromorphic parts are obtained. Section 12 gives a comparison of the present work to recent related works.

To get papers of manageable size we have separated the work into three articles, of which this is the first. The second¹⁴ contains the proof that in the pole-decomposition functions that we have separated out the contours in the angular variables Ω_i can be distorted away from all singularities, with the exception of three Feynman denominators, one from each side s of the original triangle graph G , and the end points of the radial integrations. This means that each of the distinguished contributions corresponding to a *separable* graph g is essentially the same as the usual triangle-graph function, multiplied by a bounded function of the variables r_i and Ω_i and integrated over a compact domain in these variables. The factor $U(L)$ supplies the quantum analog of the appropriate classical electromagnetic field.

The final but crucial point is then to show that the remaining parts, which are specified by compactly expressed integrals, give contributions that tend to zero in the macroscopic regime, relative to distinguished part discussed above, which meets the correspondence-principle and pole-factorization requirements. The required properties of these integrals are established in the third paper¹⁵. Our original report¹⁶ contains all three parts in one place.

2. Basic Momentum–Space Formulas

The separation of the soft–photon interaction into its quantum and classical parts is defined in Eq. (1.1). This separation is defined in a mixed representation in which hard photons are represented in coordinate space and soft photons are represented in momentum space. In this representation one can consider a “generalized propagator”. It propagates a charged particle from a hard–photon vertex y to a hard–photon vertex x with, however, the insertion of soft–photon interactions.

Suppose, for example, one inserts the interactions with two soft photons of momenta k_1 and k_2 and vector indices μ_1 and μ_2 . Then the generalized propagator is

$$\begin{aligned} P_{\mu_1, \mu_2}(x, y; k_1, k_2) &= \int \frac{d^4 p}{(2\pi)^4} e^{-ipx + i(p+k_1+k_2)y} \\ &\times \frac{i}{\not{p} - m + i0} \gamma_{\mu_1} \frac{i}{\not{p} + \not{k}_1 - m + i0} \gamma_{\mu_2} \frac{i}{\not{p} + \not{k}_1 + \not{k}_2 - m + i0}. \end{aligned} \quad (2.1)$$

The generalization of this formula to the case of an arbitrary number of inserted soft photons is straightforward. The soft–photon interaction γ_{μ_j} is separated into its parts Q_{μ_j} and C_{μ_j} by means of (1.1), with the x and y defined as in (1.3).

This separation of the soft–photon interaction into its quantum and classical parts can be expressed also directly in momentum space. Using (1.2) and (1.3), and the familiar identities

$$\frac{1}{\not{p} - m} \not{k} \frac{1}{\not{p} + \not{k} - m} = \frac{1}{\not{p} - m} - \frac{1}{\not{p} + \not{k} - m}, \quad (2.2)$$

and

$$\left(-\frac{\partial}{\partial p^\mu} \right) \frac{1}{\not{p} - m} = \frac{1}{\not{p} - m} \gamma_\mu \frac{1}{\not{p} - m}, \quad (2.3)$$

one obtains for the (generalized) propagation from y to x , with a single classical interaction inserted, the expression (with the symbol m standing henceforth for $m - i0$)

$$\begin{aligned}
P_\mu(x, y; C, k) &= \int \frac{d^4 p}{(2\pi)^4} \left(\frac{i}{\not{p} - m} \not{k} \frac{i}{\not{p} + \not{k} - m} \right) \frac{z_\mu}{zk + i0} e^{-ipz +iky} \\
&= \int \frac{d^4 p}{(2\pi)^4} \left(\frac{i}{\not{p} - m} \not{k} \frac{i}{\not{p} + \not{k} - m} \right) \frac{1}{zk + i0} \left(\frac{i\partial}{\partial p^\mu} \right) e^{-ipz +iky} \\
&= \int \frac{d^4 p}{(2\pi)^4} e^{-ipz +iky} \frac{1}{zk + i0} \left(-i \frac{\partial}{\partial p^\mu} \right) \left(\frac{i}{\not{p} - m} \not{k} \frac{i}{\not{p} + \not{k} - m} \right) \\
&= \int \frac{d^4 p}{(2\pi)^4} e^{-ipz +iky} \lim_{\epsilon \rightarrow 0} (-i) \int_0^\infty d\lambda e^{i\lambda(zk + i\epsilon)} \\
&\quad \times \left(-i \frac{\partial}{\partial p^\mu} \right) \left(\frac{i}{\not{p} - m} \not{k} \frac{i}{\not{p} + \not{k} - m} \right) \\
&= \int \frac{d^4 p}{(2\pi)^4} \lim_{\epsilon \rightarrow 0} (-i) \int_0^\infty d\lambda e^{-i(p - \lambda k)z +iky - \epsilon\lambda} \\
&\quad \times \left(-i \frac{\partial}{\partial p^\mu} \right) \left(\frac{i}{\not{p} - m} \not{k} \frac{i}{\not{p} + \not{k} - m} \right) \tag{2.4a} \\
&= \int \frac{d^4 p}{(2\pi)^4} \lim_{\epsilon \rightarrow 0} (-i) \int_0^\infty d\lambda e^{-i(p - \lambda k)z +iky - \epsilon\lambda} \\
&\quad \times \frac{\partial}{\partial p^\mu} \left(\frac{i}{\not{p} - m} - \frac{i}{\not{p} + \not{k} - m} \right) \\
&= \int \frac{d^4 p}{(2\pi)^4} \lim_{\epsilon \rightarrow 0} (-i) \int_0^\infty d\lambda (e^{-i(p - \lambda k)z} - e^{-i(p - k - \lambda k)z}) e^{-\epsilon\lambda} \\
&\quad \times e^{iky} \frac{\partial}{\partial p^\mu} \left(\frac{i}{\not{p} - m} \right) \\
&= \int \frac{d^4 p}{(2\pi)^4} \lim_{\epsilon \rightarrow 0} (-i) \int_0^1 d\lambda e^{-i(p - \lambda k)z} e^{-\epsilon\lambda} \times e^{iky} \frac{\partial}{\partial p^\mu} \left(\frac{i}{\not{p} - m} \right) \\
&= \int \frac{d^4 p}{(2\pi)^4} e^{-ipz +iky} \int_0^1 d\lambda \left(-i \frac{\partial}{\partial p^\mu} \right) \left(\frac{i}{\not{p} + \lambda \not{k} - m} \right) \tag{2.4b} \\
&= \int \frac{d^4 p}{(2\pi)^4} e^{-ipz +iky} \int_0^1 d\lambda \left(\frac{i}{\not{p} + \lambda \not{k} - m} \gamma_\mu \frac{i}{\not{p} + \lambda \not{k} - m} \right). \tag{2.4c}
\end{aligned}$$

Comparison of the result (2.4b) to (2.1) shows that the result in momentum

space of inserting a single quantum vertex j into a propagator $i(\not{p} - m)^{-1}$ is produced by the action of the operator

$$\hat{C}_{\mu_j}(k_j) = \int_0^1 d\lambda_j O(p \rightarrow p + \lambda_j k_j) \left(-i \frac{\partial}{\partial p^{\mu_j}} \right) \quad (2.5)$$

upon the propagator $i(\not{p} - m)^{-1}$ that was present *before* the insertion of the vertex j . One must, of course, also increase by k_j the momentum entering the vertex at y . The operator $O(p \rightarrow p + \lambda_j k_j)$ replaces p by $p + \lambda_j k_j$.

Suppose that there were already a soft-photon insertion on the charged $-$ particle line L so that the propagator before the insertion of vertex j were

$$P_{\mu_1}(p; k_1) = \frac{i}{\not{p} - m} \gamma_{\mu_1} \frac{i}{\not{p} + \not{k}_1 - m}. \quad (2.6)$$

And suppose the vertex j is to be inserted in all possible ways into this line (i.e., on both sides of the already-present vertex 1). Then the same argument as before, with (2.2) replaced by its generalization⁹

$$\begin{aligned} & \frac{1}{\not{p} - m} \not{k}_j \frac{1}{\not{p} + \not{k}_j - m} \gamma_{\mu_1} \frac{1}{\not{p} + \not{k}_j + \not{k}_1 - m} \\ & + \frac{1}{\not{p} - m} \gamma_{\mu_1} \frac{1}{\not{p} + \not{k}_1 - m} \not{k}_j \frac{1}{\not{p} + \not{k}_j + \not{k}_1 - m} \\ & = \frac{1}{\not{p} - m} \gamma_{\mu_1} \frac{1}{\not{p} + \not{k}_1 - m} \\ & - \frac{1}{\not{p} + \not{k}_j - m} \gamma_{\mu_1} \frac{1}{\not{p} + \not{k}_j + \not{k}_1 - m}, \end{aligned} \quad (2.7)$$

shows that the effect in momentum space is again given by the operator $\hat{C}_{\mu_j}(k_j)$ defined in (2.5).

This result generalizes to an arbitrary number of inserted classical photons, and also to an arbitrary generalized propagator: the momentum-space result of inserting in all orders into any generalized propagator $P_{\mu_1 \dots \mu_n}(p; k_1, \dots, k_n)$ a set of N classically interacting photons with $j = n + 1, \dots, n + N$ is

$$\prod_{j=n+1}^{n+N} \widehat{C}_{\mu_j}(k_j) P_{\mu_1, \dots, \mu_n}(p; k_1, \dots, k_n) = \int_0^1 \dots \int_0^1 d\lambda_{n+1} \dots d\lambda_{n+N} \prod_{j=1}^N \left(-i \frac{\partial}{\partial p^{\mu_{n+j}}} \right) P_{\mu_1, \dots, \mu_n}(p+a; k_1, \dots, k_n) \quad (2.8)$$

where $a = \lambda_{n+1}k_{n+1} + \dots + \lambda_{n+N}k_{n+N}$. The operations are commutative, and one can keep each $\lambda_j = 0$ until the integration on λ_j is performed.

To obtain the analogous result for the quantum interactions we introduce the operator $\widehat{D}_{\mu_j}(k_j)$ whose action is defined as follows:

$$\begin{aligned} \widehat{D}_{\mu_j}(k_j) \frac{i}{\not{p} - m} &= \frac{i}{\not{p} - m} \gamma_{\mu_j} \frac{i}{\not{p} + \not{k}_j - m}, \\ \widehat{D}_{\mu_j}(k_j) \frac{i}{p - m} \gamma_{\mu_1} \frac{i}{\not{p} + \not{k}_1 - m} &= \frac{i}{\not{p} - m} \gamma_{\mu_j} \frac{i}{\not{p} + \not{k}_j - m} \gamma_{\mu_1} \frac{i}{\not{p} + \not{k}_j + \not{k}_1 - m}, \\ &+ \frac{i}{\not{p} - m} \gamma_{\mu_1} \frac{i}{\not{p} + \not{k}_1 - m} \gamma_{\mu_j} \frac{i}{\not{p} + \not{k}_j + \not{k}_1 - m}, \\ &\text{etc.} \end{aligned} \quad (2.9)$$

That is, $\widehat{D}_{\mu_j}(k_j)$ acts on any generalized propagator by inserting in all possible ways an interaction with a photon of momentum k_j and vector index μ_j . Then one may define

$$\widehat{Q}_{\mu_j}(k_j) = \widehat{D}_{\mu_j}(k_j) - \widehat{C}_{\mu_j}(k_j). \quad (2.10)$$

Then the result in momentum space of inserting in all possible ways (i.e., in all possible orders) into any generalized propagator P of the kind illustrated in (2.1) a set of J quantum interactions and a set of J' classical interactions is

$$\prod_{j' \in J'} \widehat{C}_{\mu_{j'}}(k_{j'}) \prod_{j \in J} \widehat{Q}_{\mu_j}(k_j) P. \quad (2.11)$$

Consideration of (2.3) and (2.9) shows that the operators \widehat{C}_i and \widehat{Q}_i appearing in (2.11) all commute, provided we reserve until the end all integrations over the variables λ_i , in order for the action of the operators \widehat{D}_i to be well defined.

One may not wish to combine the results of making insertions in all orders. To obtain the result of inserting the classical interaction at just one place, identified by the subscript $j \in \{1, \dots, n\}$, into a (generalized) propagator $P_{\mu_1 \dots \mu_n}(p; k_1, \dots, k_n)$, abbreviated now by P_{μ_j} , one begins as in (2.4) with $k_j^{\sigma_j} P_{\sigma_j}$ in place of the quantity appearing in the bracket. However, one does not introduce (2.2), which led to the restriction of the integration to the range $1 \geq \lambda_j \geq 0$. Then, provided $k_j^2 \neq 0$, equation (2.4a) gives for the result in momentum space the result produced by the action of

$$\begin{aligned} \tilde{C}_{\mu_j}(k_j) &\equiv \\ &\int_0^\infty d\lambda_j O(p_i \rightarrow p_i + \lambda_j k_j) \left(-\frac{\partial}{\partial p^{\mu_j}} \right) \end{aligned} \quad (2.12)$$

upon $k_j^{\sigma_j} P_{\sigma_j}$.

For $k_j^2 \neq 0$ this integral converges at the upper endpoint. The indefinite integral can then be defined so that it vanishes at $\lambda = \infty$. We define $\tilde{C}_{\mu_j}(k_j)$ at $k_j^2 = 0$ by then using uniformly only the contribution from the lower endpoint $\lambda = 0$, as was entailed from the start by the initially finite value of ϵ in (2.4). (Strictly speaking, one should use a Pauli-Villars regulator to define the integral in p space—then no special treatment is needed for $k_j^2 = 0$)

To obtain a form analogous to (2.12) for the quantum interaction one may use the identity

$$\begin{aligned} k_j^{\rho_j} \int_0^\infty d\lambda_j \left(-\frac{\partial}{\partial p^{\rho_j}} \right) P_{\mu_j}(p + \lambda_j k_j) \\ = \int_0^\infty d\lambda_j \left(-\frac{\partial}{\partial \lambda_j} \right) P_{\mu_j}(p + \lambda_j k_j) \\ = P_{\mu_j}(p). \end{aligned} \quad (2.13)$$

Then the momentum-space result produced by the insertion of a quantum coupling in $P_{\mu_1 \dots \mu_n}(p; k_1, \dots, k_n) = P_{\mu_j}$ at the vertex identified by μ_j is generated by the action of

$$\tilde{Q}_{\mu_j}(k_j) \equiv (\delta_{\mu_j}^{\sigma_j} k_j^{\rho_j} - \delta_{\mu_j}^{\rho_j} k_j^{\sigma_j}) \tilde{C}_{\rho_j}(k_j) \quad (2.14)$$

upon P_{σ_j} .

An analogous operator can be applied for each quantum interaction. Thus the generalized momentum–space propagator represented by a line L of G into which n quantum interactions are inserted in a fixed order is

$$\begin{aligned}
P_{\mu_1 \dots \mu_n}(p; Q, k_1, Q, k_2, \dots Q, k_n) = \\
\prod_{j=1}^n \left[\int_0^\infty d\lambda_j (\delta_{\mu_j}^{\sigma_j} k_j^{\rho_j} - \delta_{\mu_j}^{\rho_j} k_j^{\sigma_j}) \left(-\frac{\partial}{\partial p^{\rho_j}} \right) \right] \\
\left(\frac{i}{\not{p} + \not{Q} - m} \gamma_{\sigma_1} \frac{i}{\not{p} + \not{Q} + \not{k}_1 - m} \gamma_{\sigma_2} \frac{i}{\not{p} + \not{Q} + \not{k}_1 + \not{k}_2 - m} \right. \\
\left. \dots \times \gamma_{\sigma_n} \frac{i}{\not{p} + \not{Q} + \not{k}_1 + \dots \not{k}_n - m} \right), \tag{2.15}
\end{aligned}$$

where

$$a = \lambda_1 k_1 + \lambda_2 k_2 + \dots \lambda_n k_n. \tag{2.16}$$

If some of the inserted interactions are classical interactions then the corresponding factors $(\delta_{\mu_j}^{\sigma_j} k_j^{\rho_j} - \delta_{\mu_j}^{\rho_j} k_j^{\sigma_j})$ are replaced by $(\delta_{\mu_j}^{\rho_j} k_j^{\sigma_j})$.

These basic momentum–space formulas provide the starting point for our examination of the analyticity properties in momentum space, and the closely related question of infrared convergence.

One point is worth mentioning here. It concerns the conservation of charge condition $k^\mu J_\mu(k) = 0$. In standard Feynman quantum electrodynamic this condition is not satisfied by the individual photon–interaction vertex, but is obtained only by summing over all the different positions where the photon interaction can be coupled into a graph. This feature is the root of many of the difficulties that arise in quantum electrodynamics.

Equation (2.14) shows that the conservation – law property holds for the individual *quantum* vertex: there is no need to sum over different positions. The classical interaction, on the other hand, has a form that allows one easily to sum over all possible locations along a generalized propagator, even before multiplication by k^μ . This summation converts the classical interaction to a sum of two interactions, one located at each end of the line associated with the generalized propagator. (See, for example, Eq. (7.1) below). We always perform this summation. Then the classical parts of the interaction are shifted to the hard–photon interaction points, at which $k^\mu J(k) = 0$ holds.

3. The Quantum Vertex

Suppose a single quantum interaction is inserted into a line of G . Then the associated generalized propagator is given by (2.11), (2.10), (2.9), (2.5) and (2.3):

$$\begin{aligned}
P_\mu(p; \widehat{Q}, k) &= \frac{i}{\not{p} - m} \gamma_\mu \frac{i}{\not{p} + \not{k} - m} \\
&\quad - \int_0^1 d\lambda \frac{i}{\not{p} + \lambda \not{k} - m} \gamma_\mu \frac{i}{\not{p} + \lambda \not{k} - m}.
\end{aligned} \tag{3.1}$$

The first term in (3.1) is

$$\begin{aligned}
&\frac{i}{\not{p} - m} \gamma_\mu \frac{i}{\not{p} + \not{k} - m} \\
&= -\frac{(\not{p} + m)}{p^2 - m^2} \gamma_\mu \frac{(\not{p} + \not{k} + m)}{(p + k)^2 - m^2} \\
&= -(\not{p} + m) \gamma_\mu (\not{p} + \not{k} + m) \\
&\quad \times \left(\frac{1}{p^2 - m^2} \frac{1}{2pk + k^2} - \frac{1}{2pk + k^2} \frac{1}{(p + k)^2 - m^2} \right) \\
&= -\left[\frac{-(p^2 - m^2) \gamma_\mu + (\not{p} + m)(2p_\mu + \gamma_\mu \not{k})}{(2pk + k^2)(p^2 - m^2)} \right. \\
&\quad \left. - \frac{-((p + k)^2 - m^2) \gamma_\mu + (2p_\mu + 2k_\mu - \not{k} \gamma_\mu)(\not{p} + \not{k} - m)}{(2pk + k^2)(p^2 - m^2)} \right] \\
&= -\left[\frac{2p_\mu}{(\not{p} - m)(2pk + k^2)} - \frac{2p_\mu + 2k_\mu}{(\not{p} + \not{k} - m)(2pk + k^2)} \right. \\
&\quad \left. + \frac{1}{(\not{p} - m)} \times \frac{\gamma_\mu \not{k}}{(2pk + k^2)} + \frac{\not{k} \gamma_\mu}{(2pk + k^2)} \times \frac{1}{(\not{p} + \not{k} - m)} \right], \tag{3.2}
\end{aligned}$$

where $\{\gamma_\mu, \not{p}\}_+ = 2p_\mu$ has been used, and pk represents $pk + i0$.

The second term in (3.1) can be computed from standard integral tables. Then it can be cast into a form similar to (3.2) by first considering it to be a function of the variable $t = p^2 - m^2$, with pk and k^2 regarded as parameters, next separating it into its meromorphic and nonmeromorphic parts in this variable t , and finally evaluating its meromorphic part as a sum of poles times residues.

This gives for the meromorphic part

$$\begin{aligned}
& \left[\int_0^1 d\lambda \frac{1}{\not{p} + \lambda \not{k} - m} \gamma_\mu \frac{1}{\not{p} + \lambda \not{k} - m} \right]_{Mero} \\
&= \left[\frac{(\not{p} + m) \gamma_\mu (\not{p} + m)}{2pk (p^2 - m^2)} - \frac{(\not{p} + \not{k} + m) \gamma_\mu (\not{p} + \not{k} + m)}{2(p+k)k ((p+k)^2 - m^2)} \right]_{Mero} \\
&= \frac{2p_\mu}{2pk (\not{p} - m)} - \frac{2p_\mu + 2k_\mu}{2(p+k)k (\not{p} + \not{k} - m)}, \tag{3.3}
\end{aligned}$$

where a term not depending on $(p^2 - m^2)$ has been dropped from the last line.

The singularities of this function at $pk = 0$ and $(p+k)k = 0$ are artifacts of the separation into meromorphic and non meromorphic parts: their sum does not have singularities at generic points on these surfaces. Thus we may replace pk by $pk+i0$ in both the meromorphic and non meromorphic parts and introduce the identities

$$\frac{1}{2pk} = \frac{1}{2pk + k^2} \left(1 + \frac{k^2}{2pk} \right) \tag{3.4a}$$

and

$$\frac{1}{2pk + 2k^2} = \frac{1}{2pk + k^2} \left(1 - \frac{k^2}{2pk + 2k^2} \right). \tag{3.4b}$$

Then the combination of (3.2) and (3.3) gives

$$\begin{aligned}
& P_\mu (p; Q, k)_{Mero} \\
&= \frac{1}{2pk + k^2} \left[\frac{1}{\not{p} - m} \left(\frac{2p_\mu k^2}{2pk} - \gamma_\mu \not{k} \right) \right. \\
&\quad \left. + \left(\frac{(2p_\mu + 2k_\mu)k^2}{2pk + 2k^2} - \not{k} \gamma_\mu \right) \frac{1}{\not{p} + \not{k} - m} \right]. \tag{3.5}
\end{aligned}$$

This function is of zeroth order in $|k|$, whereas the individual contributions (3.2) and (3.3) are each of order $|k|^{-1}$.

The result (3.3) can be obtained also directly by inspection of the integral appearing on the left-hand side, written in the form

$$\int_0^1 d\lambda \frac{(\not{p} + \lambda \not{k} + m) \gamma_\mu (\not{p} + \lambda \not{k} + m)}{(p^2 - m^2 + 2pk\lambda + k^2\lambda^2)^2}.$$

The singularities of this integral lying along the surface $p^2 = m^2$ arise from the endpoint $\lambda = 0$ of the domain of integration. Thus the analytic character of these

singularities is controlled by the character of the integrand in an arbitrarily small neighborhood of this endpoint. Positive powers of λ in the numerator diminish the contributions from this endpoint, and lead to singularities on $p^2 = m^2$ that are, in form, not as strong as the singularity coming from the terms that are of zeroth order in λ . Thus to find the strongest singularity we may set the λ 's appearing in the numerator to zero. For similar reasons we can set the λ^2 terms in the denominator equal to zero, provided the coefficient $2pk$ of the first power of λ is nonzero. Thus the strongest singularity of the integral arising from the lower endpoint is

$$\begin{aligned} \int_0^\infty d\lambda \frac{(\not{p} + m)\gamma_\mu(\not{p} + m)}{(p^2 - m^2 + 2pk\lambda)^2} \\ = \frac{(\not{p} + m)\gamma_\mu(\not{p} + m)}{2pk(p^2 - m^2)}. \end{aligned} \quad (3.6)$$

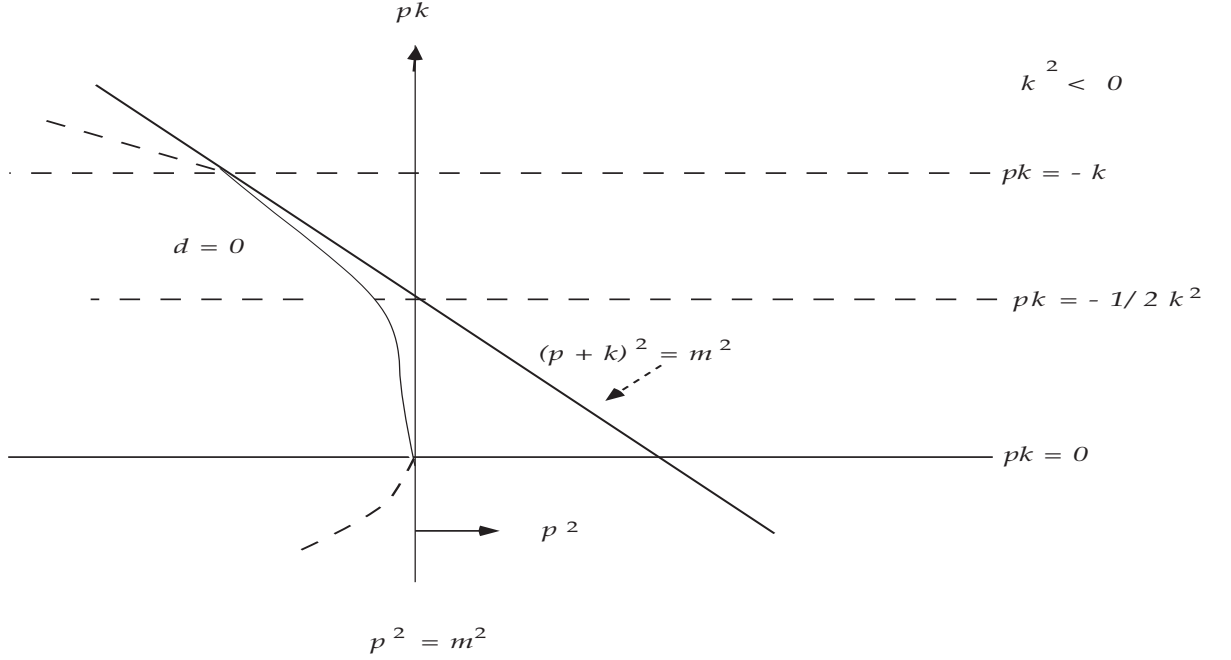
This is just the result obtained from the full calculation. The other term in (3.3) comes from the other endpoint, $\lambda = 1$. Because the strongest or dominant singularities coming from the two endpoints are poles any other singularities coming from these endpoints belong to the nonmeromorphic part.

The full nonmeromorphic part of $P_\mu(p; Q, k)$ is, by direct calculation,

$$\begin{aligned} P_\mu(p; Q, k)_{NonMero} \\ = \left[(\not{p} + m)\gamma_\mu(\not{p} + m) \left(\frac{-2k^2}{-d} \right) \right. \\ + ((\not{k}\gamma_\mu(\not{p} + m) + (\not{p} + m)\gamma_\mu \not{k}) \left(\frac{2pk}{-d} \right) \\ + \not{k}\gamma_\mu \not{k} \left(\frac{-2(p^2 - m^2)}{-d} \right) \left. \right] \\ \times \left[\frac{1}{\sqrt{-d}} \log \left(\frac{1 - \frac{\sqrt{-d}}{2pk + 2k^2}}{1 + \frac{\sqrt{-d}}{2pk + 2k^2}} \right) \right. \\ \left. - \frac{1}{\sqrt{-d}} \log \left(\frac{1 - \frac{\sqrt{-d}}{2pk}}{1 + \frac{\sqrt{-d}}{2pk}} \right) + \frac{2}{2pk + 2k^2} - \frac{2}{2pk} \right] \end{aligned} \quad (3.7)$$

where $-d = (2pk)^2 - 4k^2(p^2 - m^2) = (2(p + k)k)^2 - 4k^2((p + k)^2 - m^2)$. The two non-log terms in the final square bracket cancel the pole singularity in

Figure 1: The singularities of $P_\mu(p; Q, k)$ are confined to the surfaces $p^2 - m^2 = 0$, $(p + k)^2 - m^2 = 0$, and the branch of $d = 0$ lying between $pk = 0$ and $pk = -k^2$.



$t = p^2 - m^2$ at $d = 0$ that would otherwise arise from the small- d behavior of the log terms.

The singularity surfaces of $P_\mu(p; Q, k)$ are shown in Figure 1.

The singularities of $P_\mu(p; Q, k)$ are confined to the surfaces $p^2 - m^2 = 0$, $(p + k)^2 - m^2 = 0$, and to the portion of the surface $d = 0$ that lies between $pk = 0$ and $pk = -k^2$. Except at points of contact between two of these three surfaces the function $P_\mu(p; Q, k)$ is analytic on the three surfaces $2pk = 0$, $2pk + k^2 = 0$, and $2pk + 2k^2 = 0$, and has the form $d^{-3/2}$ on the singular branch of the surface $d = 0$. It has both pole and logarithmic singularities on the surfaces $p^2 - m^2 = 0$ and $(p + k)^2 - m^2 = 0$. The $i0$ rule associated with $d = 0$ matches the $i0$ rules at $p^2 = m^2$ and $(p + k)^2 = m^2$ at their points of contact.

The meromorphic and nonmeromorphic parts of $P_\mu(p; Q, k)$ each separately have singularities on the surfaces $2pk = 0$, $2pk + k^2 = 0$ and $2pk + 2k^2 = 0$.

The results of this section may be summarized as follows: the insertion of a single quantum interaction into a propagator $i(\not{p} - m)^{-1}$ associated with G converts it into a sum of three terms. The first is a propagator $i(\not{p} - m)^{-1}$ multiplied by a factor that is zeroth order in $r = |k|$. The second is a propagator $i(\not{p} + \not{k} - m)^{-1}$ multiplied by a factor that is zeroth order in r . The third is a vertex-type term, which has logarithmic singularities on the two surfaces $p^2 - m^2 = 0$ and $(p + k)^2 - m^2 = 0$. This latter term has a typical vertex-correction type of analytic structure even though it is represented diagrammatically as (the nonmeromorphic part of) a simple vertex insertion.

4. Triangle–Diagram Process

In the introduction we described a hard–photon process associated with a triangle graph G . In this section we describe the corrections to it arising from a single soft photon that interacts with G in the way shown in Figure 2.

Each external vertex v_i of Fig. 1 represents the two vertices upon which the two external hard photons are incident, together with the charged–particle line that runs between them. The momenta of the various external photons can be chosen so that the momentum–energy of this connecting charged–particle line is far from the mass shell, in the regime of interest. In this case the associated propagator is an analytic function. We shall, accordingly, represent the entire contribution associated with each external vertex v_i by the single symbol V_i , and assume only that the corresponding function is analytic in the regime of interest. The analysis will then cover also cases outside of quantum–electrodynamics.

In Fig. 2 the two solid lines with Q–vertex insertions represent generalized propagators. We consider first the contributions that arise from the meromorphic or pole contributions to these two generalized propagators.

Each generalized propagator has, according to (3.5), two pole contributions, one proportional to the propagator $i(\not{p} - m)^{-1}$, the other proportional to $i(\not{p} + \not{k} - m)^{-1}$. This gives four terms, one corresponding to each of the four graphs in Fig. 3. Each line of Fig. 3 represents a propagator $i(\not{p}_i - m)^{-1}$ or $i(\not{p}_i + \not{k} - m)^{-1}$, with $i = 1$ or 2 labelling the two relevant lines. The singularities on the Landau triangle–diagram surface $\varphi = 0$ arise from a conjunction of three such singularities, one from each side of the triangle in Figure 3.

The graph (a) represents, by virtue of (3.5), the function

$$\begin{aligned}
 F_a = & \int \frac{d^4 p}{(2\pi)^4} \int_{|k| \leq \delta} \frac{d^4 k}{(2\pi)^4} i(k^2 + i0)^{-1} \\
 & \text{Tr} \left\{ \frac{i(\not{p} + m)}{p^2 - m^2} V_1 \frac{(\not{p}_1 + m)}{p_1^2 - m^2} \left(\frac{2p_{1\mu} k^2 (2p_1 k)^{-1} - \gamma_\mu \not{k}}{2p_1 k + k^2} \right) \right. \\
 & \left. V_2 \left(\frac{2p_{2\mu} k^2 (2p_2 k)^{-1} - \not{k} \gamma_{\mu 2}}{2p_2 k + k^2} \right) \frac{(\not{p}_2 + m)}{p_2^2 - m^2} V_3 \right\} \quad (4.1)
 \end{aligned}$$

where $p_1 = p + q_1$, $p_2 = p - q_3$, $p_i k = p_i k + i0$, and q_i is the momentum–energy carried out of vertex v_i by the external hard photons incident upon it. The vector $p \equiv p_3$ is the momentum–energy flowing along the internal line that runs

Figure 2: Graph representing a soft-photon correction to a hard-photon triangle-diagram process. Hard and soft photons are represented by dashed and wiggly lines, respectively.

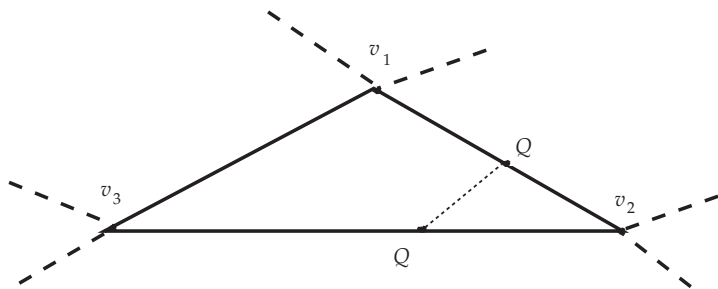
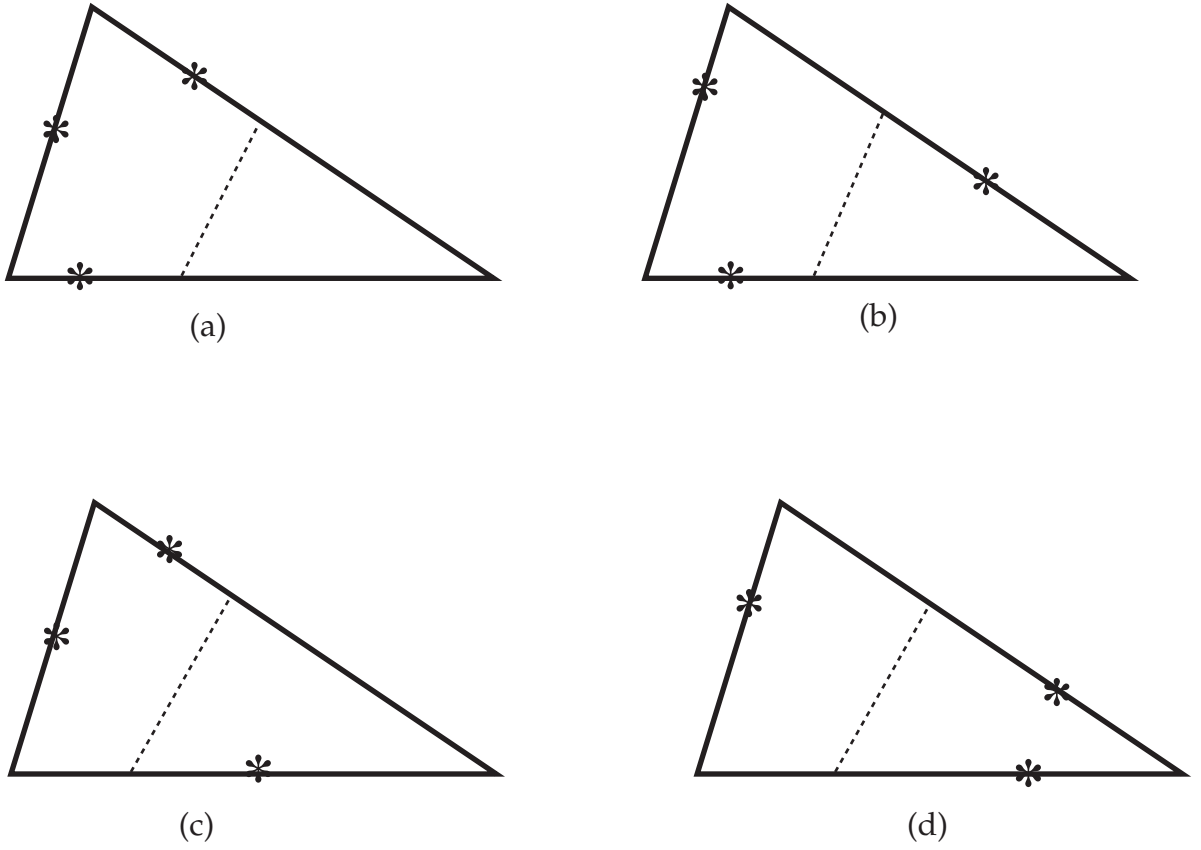


Figure 3: Graphs representing the four contributions that arise from inserting into each of the two generalized propagators represented in Fig. 2 the sum of the two meromorphic terms given by (3.5).



from v_1 to v_3 .

To give meaning to the function $(k^2 + i0)^{-1}$ at the point $k = 0$ we introduce polar coordinates, $k = r\Omega$, and write

$$\int_{|k| \leq \delta} \frac{d^4 k}{k^2 + i0} f(k) = \int_0^\delta 2r dr \int d^4 \Omega \frac{\delta(\Omega_0^2 + \vec{\Omega}^2 - 1)}{\Omega^2 + i0} f(r\Omega) \quad (4.2)$$

Then F_a becomes

$$\begin{aligned} F_a = & \int \frac{d^4 p}{(2\pi)^4} \int_0^\delta 2r dr \int \frac{d^4 \Omega}{(2\pi)^4} \frac{i\delta(\Omega_0^2 + \vec{\Omega}^2 - 1)}{\Omega^2 + i0} \\ & \text{Tr} \left\{ \frac{i(\not{p} + m)}{p^2 - m^2} V_1 \frac{(\not{p}_1 + m)}{p_1^2 - m^2} \left(\frac{2p_{1\mu} \Omega^2 (2p_1 \Omega)^{-1} - \gamma_\mu \not{\Omega}}{2p_1 \Omega + r\Omega^2} \right) \right. \\ & \left. V_2 \left(\frac{2p_{2\mu} \Omega^2 (2p_2 \Omega)^{-1} - \not{\Omega} \gamma_\mu}{2p_2 \Omega + r\Omega^2} \right) \frac{(\not{p}_2 + m)}{p_2^2 - m^2} V_3 \right\}. \end{aligned} \quad (4.3)$$

where $p_i \Omega$ represents $p_i \Omega + i0$.

The integrand of this function behaves near $r = 0$ like $r dr$. Hence the integral is infrared finite.

We are interested in the form of the singularity at interior points of the positive- α branch of the Landau triangle-diagram surface $\varphi = 0$. Let $q = (q_1, q_2, q_3)$ be such a point on $\varphi(q) = 0$. The singularity at q is generated by the pinching of the contour of integration in p -space by the three surfaces $p_i^2 - m^2 = 0$. This pinching occurs at a point in the domain of integration where the three vectors (p_1, p_2, p_3) lie at a point $(p_1(q), p_2(q), p_3(q))$ that is determined uniquely by the value q on $\varphi = 0$. At this point none of these vectors is parallel to any other one. Consequently, in view of the $i0$ rules described in connection with Fig. 1., it is possible, in a sufficiently small p -space neighborhood of $(p_1(q), p_2(q), p_3(q))$, for sufficiently small δ , to shift the contour of integration in Ω space simultaneously into the regions $\text{Im } p_1 \Omega > 0$ and $\text{Im } p_2 \Omega > 0$, and to make thereby the denominator factors $p_i \Omega$ and $p_i \Omega + r\Omega^2$, for $i \in \{1, 2\}$, all simultaneously nonzero, for all points on the Ω contour. In this way the factors in (4.3) that contain these denominator functions can all be made analytic in all variables in a full neighborhood of the pinching point. Consequently, these factors can, for the purpose of examining the character of the singularity along $\varphi = 0$ be incorporated into the analytic factor V_2 .

The computation of the form of the singularity on $\varphi = 0$ then reduces to the usual one: the singularity has the form $\log \varphi$, and the discontinuity is given by the Cutkosky rule, which instructs one to replace each of the three propagator-poles $i(p_i^2 - m^2)$ by $2\pi\delta(p_i^2 - m^2)$.

This gives most of what we need in this special case: it remains only to be shown that the remaining singularities on $\varphi = 0$ are weaker in form than $\log \varphi$.

If one were to try to deal in the same way with the function represented by Fig. 2, but with the original vertices γ_μ rather than Q_μ , then (3.2) would be used instead of (3.5) and the integration over r in the expression replacing (4.3) would become infrared divergent. The definition of $k^2 + i0$ embodied in (4.2) is insufficient in this case. A proper treatment¹⁰ shows that the dominant singularity on the surface $\varphi = 0$ would in this case be $(\log \varphi)^2$.

The graph (b) of Fig. 3 represents the function

$$\begin{aligned}
F_b = & \int \frac{d^4 p}{(2\pi)^4} \int_0^\delta 2r dr \int \frac{d^4 \Omega}{(2\pi)^4} \frac{i\delta(\Omega_0^2 + \vec{\Omega}^2 - 1)}{\Omega^2 + i0} \\
& \text{Tr} \left\{ \frac{i(\not{p} + m)}{p^2 - m^2} V_1 \left(\frac{(2p_{i\mu} + 2r\Omega_\mu)\Omega^2(2p_1\Omega + 2r\Omega^2)^{-1} - \not{p}\gamma_\mu}{2p_1\Omega + r\Omega^2} \right) \right. \\
& \times \frac{(\not{p}_1 + r\not{\Omega} + m)}{(p_1 + r\Omega)^2 - m^2} V_2 \\
& \left. \times \left(\frac{2p_{2\mu}\Omega^2(2p_2\Omega) - \gamma_\mu \not{\Omega}}{2p_2\Omega + r\Omega^2} \right) \frac{(\not{p}_2 + m)}{p_2^2 - m^2} V_3 \right\} \quad (4.4)
\end{aligned}$$

where $p_i\Omega$ represents $p_i\Omega + i0$. This integral also is free of infrared divergences. It is shown in ref. 15 that its singularity on $\varphi = 0$ has the form $\varphi^2 \log \varphi$. The same result is obtained for graphs (c) and (d) of Fig. 3.

The remaining contributions to the process represented in Fig. 2 involve the nonmeromorphic parts of at least one of the two generalized propagators. These nonmeromorphic parts are given by (3.7). This expression gives logarithmic singularities on $p_i^2 - m^2 = 0$ and $(p_i + r\Omega)^2 - m^2 = 0$, for $i = 1$ and 2 . It gives singularities also on $p_i\Omega = 0$ and $p_i\Omega + r\Omega^2 = 0$, and a $d_i^{-3/2}$ singularity on the portion of the surface $d_i = 0$ that lies between $p_i\Omega = 0$ and $p_i\Omega + r\Omega^2 = 0$.

For p in a small neighborhood of the fixed pinching point one can again, for sufficiently small δ , distort the Ω contour simultaneously into the upper-half

planes of both $p_1\Omega$ and $p_2\Omega$, and thereby avoid simultaneously the zeros of $p_i\Omega$, $p_i\Omega = r\Omega^2$, and also those of

$$d_i(2r)^{-2} = (p_i\Omega)^2 - (p_i^2 - m^2)\Omega^2.$$

Thus for every point on the Ω contour the nonmeromorphic part of the propagator associated with line i takes, near the pinching point, the form

$$A_i \frac{1}{r} \log \frac{(p_i + r\Omega)^2 - m^2}{p_i^2 - m^2}, \quad (4.5)$$

where A_i is analytic in all variables.

If we combine the two factors (4.5), one from each end of the photon line, then the two displayed powers of r^{-1} join with rdr to give dr/r . Consequently, if each of the two logarithmic factors in (4.5) were treated separately then an infrared divergence would ensue. However, the entire (4.5), taken as a unit, is of zeroth order in r , and it gives no such divergence. It is therefore necessary in the treatment of the nonmeromorphic part to keep together those contributions coming from various logarithmic singularities, such as the two logarithmic singularities of (4.5), that are naturally tied together by a cut. By contrast, in the meromorphic part it was possible to treat separately the contributions from the two different pole singularities associated with each of the two sides $i = 1$ and $i = 2$ of the triangle: for the meromorphic part each of the four terms indicated in Fig. 3 is separately infrared convergent.

The product of the two factors (4.5) gives an integrand factor of the form

$$I = \frac{dr}{r} \left(\log \frac{(p_1 + r\Omega)^2 - m^2}{p_1^2 - m^2} \right) \left(\log \frac{(p_2 + r\Omega)^2 - m^2}{p_2^2 - m^2} \right). \quad (4.6)$$

The dominant singularity on $\varphi = 0$ generated by this combination is shown in ref. 15 to be of the form $\varphi^2(\log \varphi)^2$. If one combines the nonmeromorphic part from one end of the soft-photon line with the meromorphic part from the other end then the resulting dominant singularity on $\varphi = 0$ has the form $\varphi \log \varphi$. Replacement of *one* of the two Q -type interactions in Fig. 2 by a C -type interaction does not materially change things. The results are described in ref. 15.

We now turn to the generalization of these results to processes involving arbitrary numbers of soft photons, each having a Q -type interaction on at least one end.

5. Residues of Poles in Generalized Propagators

Consider a generalized propagator that has only quantum–interaction insertions. Its general form is, according to (2.15),

$$\prod_{j=1}^n \left[\left(\delta_{\mu_j}^{\sigma_j} k_j^{\rho_j} - \delta_{\mu_j}^{\rho_j} k_j^{\sigma_j} \right) \int_0^\infty d\lambda_j \left(-\frac{\partial}{\partial p^{\rho_j}} \right) \right] \\ \left(\frac{i}{\not{p} + \not{a} - m} \gamma_{\sigma_1} \frac{i}{\not{p} + \not{a} + \not{k}_1 - m} \gamma_{\sigma_2} \frac{i}{\not{p} + \not{a} + \not{k}_1 + \not{k}_2 - m} \right. \\ \left. \cdots \times \gamma_{\sigma_n} \frac{i}{\not{p} + \not{a} + \not{k}_1 \cdots + \not{k}_n - m} \right) \quad (5.1)$$

where

$$a = \lambda_1 k_1 + \cdots + \lambda_n k_n. \quad (5.2)$$

The singularities of (5.1) that arise from the multiple end–point $\lambda_1 = \lambda_2 = \cdots \lambda_n = 0$ lie on the surfaces

$$p_i^2 = m^2, \quad (5.3)$$

where now (in contrast to earlier sections)

$$p_i = p + k_1 + k_2 + \cdots + k_i. \quad (5.4)$$

At a point lying on only one of these surfaces the strongest of these singularities is a pole. As the first step in generalizing the results of the preceding section to the general case we compute the residues of these poles.

The Feynman function appearing in (5.1) can be decomposed into a sum of poles times residues. At the point $a = 0$ this gives

$$\frac{i(\not{p} + m) \gamma_{\mu_1} i(\not{p} + \not{k}_1 + m) \gamma_{\mu_2} \cdots \gamma_{\mu_n} i(\not{p} + \cdots + \not{k}_n + m)}{(p^2 - m^2)((p + k_1)^2 - m^2) \cdots ((p + \cdots + k_n)^2 - m^2)} \\ = \sum_{i=0}^n \frac{N_{1i}}{D_{1i}} \frac{i(\not{p}_i + m)}{p_i^2 - m^2} \frac{N_{2i}}{D_{2i}}, \quad (5.5)$$

where for each i the numerator occurring on the right–hand side of this equation is identical to the numerator occurring on the left–hand side. The denominator factors are

$$D_{1i} = \prod_{j < i} (2p_i k_{ij} + (k_{ij})^2 + i0), \quad (5.6a)$$

and

$$D_{2i} = \prod_{j>i} (2p_i k_{ij} + (k_{ij})^2 + i0), \quad (5.6b)$$

where

$$k_{ij} = \sigma_{ij}[(k_1 + \cdots + k_j) - (k_1 + \cdots + k_i)]. \quad (5.7)$$

The sign $\sigma_{ij} = \pm$ in (5.7) is specified in the following way: in order to make the pole-residue formula well defined each quantity $p_s k_i$ is replaced by $p_s k_i + i\epsilon_i$ with $\epsilon_i \gg \epsilon_{i+1} > 0$, for the ordering (6.1). Thus each ϵ_i is taken to be much larger than the next one, so it that it dominates over any sum of smaller ones. This makes each difference of denominators that occurs in the pole-residue decomposition well defined, with a well-defined nonvanishing imaginary part. Then the sign σ_{ij} in (5.7), is fixed so as to make the imaginary part of the (i, j) factor in (5.6) positive. Then the limit where all $\epsilon_i \rightarrow 0$ is concordant with (5.6).

Since the singularities in question arise from the multiple endpoint $\lambda_1 = \cdots \lambda_n = 0$ it is sufficient for the determination of the analytic character of the singularity to consider an arbitrarily small neighborhood of this endpoint. We shall consider, for reasons that will be explained later, only points in a closed domain in the variables k_j upon which the parameters $p_i k_j$ and $2p_i k_{ij} + (k_{ij})^2$ are all nonzero. Then the factors D_{1i}^{-1} and D_{2i}^{-1} are analytic functions of the variables λ_j in a sufficiently small neighborhood of the point $\lambda_1 = \cdots = \lambda_n = 0$. Hence a power series expansion in these variables can be introduced.

The dominant singularity coming from the multiple end point $\lambda_1 = \cdots = \lambda_n = 0$ is obtained by setting to zero all the λ_j coming from either the numerators N_{1i} and N_{2i} or the power series expansion of the factors D_{1i}^{-1} and D_{2i}^{-1} . Then the only remaining λ_j 's are those in the pole factor $((p_i + a)^2 - m^2)^{-1}$ itself.

Consider, then, the term in (5.1) coming from the i th term in (5.5). And consider the action of the first operator, $j = 1$, in (5.1). This integral is essentially the one that occurred in section 3. Comparison with (2.3), (3.6), and (3.3) shows that the dominant singularity on $p_i^2 - m^2 = 0$ is the function obtained by simply making the replacement

$$\int_0^\infty d\lambda_j \left(-\frac{\partial}{\partial p^{\rho_j}} \right) (O(p \rightarrow p + \lambda_j k_j)) \rightarrow p_{i\rho_j} (p_i k_j)^{-1}. \quad (5.8)$$

Each value of j can be treated in this way. Thus the dominant singularity of the generalized propagator (5.1) on $p_i^2 - m^2 = 0$ is

$$\prod_{j=1}^n \left[\left(\delta_{\mu_j}^{\sigma_j} k_j^{\rho_j} - \delta_{\mu_j}^{\rho_j} k_j^{\sigma_j} \right) p_{i\rho_j} (p_i k_j)^{-1} \right] \times \frac{N_{1i} i(\not{p}_i + m) N_{2i}}{D_{1i}(p_i^2 - m^2) D_{2i}}. \quad (5.9)$$

The numerator in (5.9) has, in general, a factor

$$\begin{aligned} & i(\not{p}_i - \not{k}_i + m) \gamma_{\sigma_i} i(\not{p}_i + m) \gamma_{\sigma_{i+1}} i(\not{p}_i + \not{k}_{i+1} + m) \\ &= i(\not{p}_i - \not{k}_i + m) \gamma_{\sigma_i} i(\not{p}_i + m) i(2p_{i\sigma_{i+1}} + \gamma_{\sigma_{i+1}} \not{k}_{i+1}) \\ & \quad + i(\not{p}_i - \not{k}_i + m) \gamma_{\sigma_i} \gamma_{\sigma_{i+1}} (p_i^2 - m^2) \\ &= i(2p_{i\sigma_i} - \not{k}_i \gamma_{\sigma_i}) i(\not{p}_i + m) i(2p_{i\sigma_{i+1}} + \gamma_{\sigma_{i+1}} \not{k}_{i+1}) \\ & \quad + i(p_i^2 - m^2) \gamma_{\sigma_i} (2p_{i\sigma_{i+1}} + \gamma_{\sigma_{i+1}} \not{k}_{i+1}) \\ & \quad + i(\not{p}_i - \not{k}_i + m) \gamma_{\sigma_i} \gamma_{\sigma_{i+1}} (p_i^2 - m^2) \end{aligned} \quad (5.10)$$

The last two terms in the last line of this equation have factors $p_i^2 - m^2$. Consequently, they do not contribute to the residue of the pole at $p_i^2 - m^2 = 0$. The terms in (5.10) with a factor $2p_{i\sigma_{i+1}}$, taken in conjunction with the factor in (5.9) coming from $j = i + 1$, give a dependence $2p_{i\rho_j} 2p_{i\sigma_j}$. This dependence upon the indices ρ_j and σ_j is symmetric under interchange of these two indices. But the other factor in (5.9) is antisymmetric. Thus this contribution drops out. The contribution proportional to $p_{i\sigma_i}$ drops out for similar reasons.

Omitting these terms that do not contribute to the residue of the pole at $p_i^2 - m^2$ one obtains in place of (5.10) the factor

$$(-i \not{k}_i \gamma_{\sigma_i}) i(\not{p}_i + m) (i \gamma_{\sigma_{i+1}} \not{k}_{i+1}) \quad (5.11)$$

which is first-order in both \not{k}_i and \not{k}_{i+1} .

The above argument dealt with the case in which $i \neq 0$ and $i \neq n$: i.e., the propagator i is neither first nor last. If $i = 0$ then there is no factor $k_i = k_0$ in (5.11): in fact no such k_j is defined. If $i = n$ then there is no factor $k_{i+1} = k_{n+1}$ in (5.11): in fact no such k_j is defined in the present context. Thus one or the

other of the two k -dependent factors drops out if propagator i is the first or last one in the sequence.

This result (5.11) is the generalization to the case $n > 1$ of the result for $n = 1$ given in (3.5). To obtain the latter one must combine (5.11) with (5.9). The effect of (5.11) is to provide, in conjunction with these pole singularities, a “convergence factor” for the factors lying on either side of each pole factor in the pole-residue decomposition (5.5). That these “convergence factors” actually lead to infrared convergence is shown in the following sections.

6. Infrared Finiteness of Scattering Amplitudes.

Let G be a hard-photon graph. Let g be a graph obtained from it by the insertion of soft photons. In this section we suppose that each soft photon is connected on both ends into G by a Q -type interaction.

Each charged-particle line segment L of G is converted into a line L' of g by the insertion of $n \geq 0$ soft-photon vertices. The line L' of g represents a generalized propagator. Let the symbols L_i , with $i \in \{0, \dots, n\}$, represent the various line segments of L' .

In this section we shall be concerned only with the contributions coming from the pole parts of the propagator described in section 5. In this case each generalized propagator is expressed by (5.9) as a sum of pole terms, each with a factorized residue enjoying property (5.11).

One class of graphs is of special interest. Suppose for each charged line L' of g there is a segment L_i such that the cutting of each of these segments L_i , together perhaps with the cutting of some hard-photon lines, separates the graph g into a set of disjoint subgraphs each of which contains precisely one vertex of the original graph G . In this case the soft-photon part of the computation decomposes into several independent parts: all dependence on the momentum k_j of the soft photon j is confined to the functional representation of the subgraph in which the line representing this photon is contained.

The purpose of this section is first to prove infrared convergence for the special case of *separable* graphs defined by two conditions. The first condition is that the graph g separate into subgraphs in the way just described. We then consider for each line L' of g a single term in the corresponding generalized propagator (5.9). The second condition is that in this term of (5.9) the factor $i/(p_i + m)(p_i^2 - m^2)^{-1}$ correspond to the line segment of L_i that is cut to produce the separation into subgraphs. Then each subgraph will contain, for each charged-particle line that either enters it or leaves it, a half-line h that contains either the set of vertices $j \geq i$, or, alternatively, the set of vertices $j < i$, of that charged-particle line.

It is also assumed that the graph G is simple: at most one line segment (i.e., edge) connects any pair of vertices of G .

The contributions associated with graphs of this kind are expected to give

the dominant singularities of the full function on the Landau surface associated with D . If the functions associated with all the various subgraphs are well defined when the momenta associated with all lines of D are placed on-mass-shell then the discontinuity of the full function across this Landau surface will be a product of these well defined functions. By virtue of the spacetime fall-off properties established in paper I these latter functions can then be identified with contributions to the scattering functions for processes involving charged external particles. The purpose of this section is to prove the infrared finiteness of these contributions to the scattering functions.

Each subgraph can be considered separately. Thus it is convenient to introduce a new labelling of the set of, say, n soft photons that couple into the subgraph under consideration. To do this the domain of integration $0 \leq |k_j| \leq \delta$, $j \in \{1, \dots, n\}$, is first decomposed into $n!$ domains according to the relative sizes of the Euclidean magnitudes $|k_j|$. Then in each of these separate domains the vectors k_i are labelled so that $|k_1| \geq |k_2| \geq \dots \geq |k_n| \geq 0$. A generalized polar coordinate system is then introduced:

$$\begin{aligned} k_1 &= r_1 \Omega_1 \\ k_2 &= r_1 r_2 \Omega_2 \\ &\vdots \\ k_n &= r_1 r_2 \cdots r_n \Omega_n. \end{aligned} \tag{6.1}$$

Here $|r_1| \leq \delta$, and $|r_j| \leq 1$ for $j = 2, \dots, n$, and $\Omega \tilde{\Omega} \equiv (\Omega_{j0})^2 + (\vec{\Omega}_j)^2 = 1$.

The factors in $D_i(a=0)$, as defined in (5.6), are $2p_i k_{ij} + (k_{ij})^2$. However, the k_{ij} are no longer given by (5.7). With our new labelling the formula (5.7) becomes

$$k_{ij} = \sum_{j' \in J(i,j)} \pm k_{j'}, \tag{6.2}$$

where the signs \pm are the same as the signs in (5.7): only the labelling of the vectors is changed.

Let $j(i, j)$ be the smallest number in the set of numbers $J(i, j)$. Then singling out this term in k_{ij} one may write

$$2p_i k_{ij} + (k_{ij})^2 = r_1 r_2 \cdots r_{j(i,j)} (2p_i \Omega_{j(i,j)} + R) \tag{6.3}$$

where R is bounded.

The zeros of the factors $(2p_i\Omega_{j(i,j)} + R)$ play an important role in the integration over Ω space. However, our objective in this section is to prove the convergence of the integrations over the radial variables r_j , under the condition that the Ω contours can be distorted so as to keep all of these Ω -dependent factors finite, and hence analytic. The validity of this distortion condition is discussed in Section 8, and proved in ref. 14.

To prove infrared convergence under this condition it is sufficient to show, for each value of j , that if the differential dr_j is considered to be of degree one in r_j then the full integrand, including the differential dr_j , is of degree at least two in r_j . This will ensure that the integration over r_j is convergent near $r_j = 0$.

The power counting in the variables $r_{j'}$ is conveniently performed in the following way: the factor $|k_j| d|k_j|$ arising from $d^4k_j/k_j^2 + i0$ gives, according to (6.1), a factor that has, in each variable $r_{j'}$, the degree of $(r_1 \cdots r_j)^2$. This factor may be separated into two factors $(r_1 \cdots r_j)$, one for each end of the photon line. Then each individual generalized propagator can be considered separately: for each coupling of a photon j carrying momentum $k_j = r_1 \cdots r_j \Omega_j$ into a half-line h we assign to h one of the two factors $(r_1 \cdots r_j)$ mentioned above. Thus each half-line h will have one such numerator factor for each of the photon lines that is incident upon it, and this numerator factor can be associated with the vertex upon which the photon line is incident. On the other hand, (6.3) entails that there is a dominator factor $r_1 \cdots r_{j(i,j)}$ associated with the j th interval of h . Finally, if the photon incident upon the endpoint of h that stands next to the interval that was cut is labelled by e then there is an extra numerator factor $r_1 \cdots r_e$: it comes from the factor k_{i+1} (or k_i) in (5.11).

We shall now show that these various numerator and denominator factors combine to produce for each j , and for each half-line upon which the soft photon j , is incident, a net degree in r_j of at least one, and for every other half-line a net degree of at least zero.

Consider any fixed j . To count powers of r_j we first classify each soft photon j' as “nondominant” or “dominant” according to whether $j' \geq j$ or $j' < j$. Any line segment of h along which flows the momentum $k_{j'}$ of a dominant photon j' will, according to (6.3), not contribute a denominator factor r_j .

Thus the denominator factors that do contribute a power of r_j can be displayed graphically by first considering the line h that starts at the initial vertex $j = e$, which stands, say, just to the right of the cut line-segment L_i , and that runs to the right. Soft photons are emitted from the succession of vertices on h , and some of these photons can be reabsorbed further to the right on h . In such cases the part of h that lies to the right of the vertex where a dominant photon is emitted but to the left of the point where it is reabsorbed may be contracted to a point: according to (6.3) none of these contracted line segments of h carry a denominator factor of r_j . If a dominant soft photon is emitted but is never reabsorbed on h then the entire part of the line h lying to the right of its point of emission can be contracted to this point.

If the line obtained by making these two changes in h is called h' then, by virtue of (6.3), there is exactly one denominator factor r_j for each line segment of h' .

Self-energy and vertex corrections are to be treated in the usual way by adding counterterms. Thus self-energy-graph insertions and vertex-correction graphs should be omitted: the residual corrections do not affect the power counting. This means that every vertex on h' , excluding the last one on the right end, will be either:

1. An original vertex from which a single nondominant photon is either emitted or absorbed; or
2. A vertex formed by a contraction. Any vertex of the latter type must have at least two nondominant soft photons connected to it, due to the exclusion of self-energy and vertex corrections.

The first kind of vertex will contribute one power of r_j to the numerator, whereas the second kind of vertex will contribute at least two powers of r_j .

Every line segment of h' has a vertex standing immediately to its left. Thus each denominator power of r_j will be cancelled by a numerator power associated with this vertex. This cancellation ensures that each half-line will be of degree at least zero in r_j .

If the soft-photon e incident upon the left-hand end of h' is nondominant then one extra power of r_j will be supplied by the factor k_e coming from (5.11).

If the soft photon e is dominant then there are two cases: either the left-most vertex of h' is the only vertex on h' , in which case there are no denominator factors of r_j , but at least one numerator factor for each k_j vertex incident on h ; or the left most vertex of h' differs from the rightmost one, and is formed by contraction, in which case at least two nondominant lines must be connected to it. These two lines deliver two powers of r_j to the numerator and hence the extra power needed to produce degree one in r_j .

This result for the individual half lines means that for the full subgraph the degree in r_j is at least one for every j . Hence the function is infrared convergent.

The argument given above covers specifically only the special class of separable graphs g . However, the argument applies essentially unchanged to the general case. The restriction to separable graphs fixed the directions that the photon loops flowed along the half-line h under consideration: each photon loop i incident upon h flowed *away* from the pole line-segment s that lies on one end of h . This entails that for any line segment j lying in h the associated denominator function f_j contains a term $2p_s k_i$ if and only if the following condition is satisfied: exactly one end of the photon loop i that carries momentum k_i is incident upon the half-line h in the interval lying between the (open) segment j and the (open) segment s that lies on the end of h .

This key property of f_j follows in general, however, directly from the formula

$$\begin{aligned}
f_j &= \sigma_{js}(\Sigma_j^2 - \Sigma_s^2) \\
&= \sigma_{js}(\Sigma_j + \Sigma_s)(\Sigma_j - \Sigma_s) \\
&= \sigma_{js}(2p_s(\Sigma_j - \Sigma_s) + \Sigma_j^2 - \Sigma_s^2),
\end{aligned} \tag{6.4}$$

where $\Sigma_j = p_s + K_j$ and $\Sigma_s = p_s + K_s$. The difference $K_j - K_s$ consists, apart from signs, of the sum of the k_i associated with the photon loops i that are incident upon h precisely *once* in the interval between the segments j and s . This entails the key property that was obtained in the separable case from the separability condition, which is consequently not needed: the arguments in this section pertaining to the powers of the r_i cover also the non-separable case.

7. Inclusion of the Classical Interactions

The power-counting arguments of the preceeding section dealt with processes containing only Q -type interactions. In that analysis the order in which these Q -type interactions were inserted on the line L of G was held fixed: each such ordering was considered separately.

In this section the effects of adding C -type interaction are considered. Each C -type interactions introduces a coupling $k^\sigma \gamma_\sigma = \not{k}$. Consequently, the Ward identities, illustrated in (2.7), can be used to simplify the calculation, but only if the contributions from all orders of its insertion are treated together. This we shall do. Thus for C -type interactions it is the operator \hat{C} defined in (2.5) that is to be used rather than the operator \tilde{C} defined in (2.12).

Consider, then, the generalized propagator obtained by inserting on some line L of G a set of n interactions of Q -type, placed in some definite order, and a set of N C -type interactions, inserted in all orders. The meromorphic part of the function obtained after the action of the n operators \tilde{Q}_j is given by (5.9). The action upon this of the N operators \hat{C}_j of (2.5) is obtained by arguments similar to those that gave (5.9), but differing by the fact that (2.5) acts upon the propagator present *before* the action of \tilde{C}_j , and the fact that now both limits of integration contribute, thus giving for each \hat{C}_j two terms on the right-hand side rather than one. Thus the action of N such \hat{C}_j 's gives 2^N terms:

$$\begin{aligned}
& \left[\prod_{j=n+1}^{n+N} \hat{C}_{\mu_j}(k_j) P_{\mu_1 \dots \mu_n}(p; Q, k_1, Q, k_2, \dots Q, k_n) \right]_{Mero} \\
&= \sum_{\Theta=1}^{2^N} Sgn(\Theta) \sum_{i=0}^n \prod_{j=n+1}^{n+N} \left(\frac{i p_{i\mu_j}^\Theta}{p_i^\Theta k_j} \right) \\
&\times \left\{ \prod_{j=1}^n \left[\left(\delta_{\mu_j}^{\sigma_j} k_j^{\rho_j} - \delta_{\mu_j}^{\rho_j} k_j^{\sigma_j} \right) \left(\frac{p_{i\rho_j}^\Theta}{p_i^\Theta k_j} \right) \right] \right\} \\
&\times \frac{N_{1i}^\Theta}{D_{1i}^\Theta} \frac{i(\not{p}_i^\Theta + m)}{(p_i^\Theta)^2 - m^2} \frac{N_{2i}^\Theta}{D_{2i}^\Theta}, \tag{7.1}
\end{aligned}$$

where

$$\Theta = (\Theta_{n+1}, \dots, \Theta_{n+N}),$$

$$\Theta_j = +1 \text{ or } 0,$$

$$\begin{aligned}
Sgn(\Theta) &= (-1)^{\Theta_{n+1}}(-1)^{\Theta_{n+2}} \dots (-1)^{\Theta_{n+N}} \\
p_i^\Theta &= p_i + \Theta_{n+1}k_{n+1} + \dots + \Theta_{n+N}k_{n+N}, \\
p_i &= p + k_1 + \dots + k_i,
\end{aligned} \tag{7.2}$$

and the superscript Θ on the N 's and D 's means that the argument p_i appearing in (5.5) and (5.6) is replaced by p_i^Θ . Note that even though the action of \hat{C}_j and \tilde{Q}_j involve integrations over λ and differentiations, the meromorphic parts of the resulting generalized propagators are expressed by (7.1) in relatively simple closed form. These meromorphic parts turn out to give the dominant contributions in the mesoscopic regime, as we shall see.

The essential simplification obtained by summing over all orders of the C -type insertions is that after this summation each C -type interaction gives just two terms. The first term is just the function before the action of \hat{C}_j multiplied by $ip_{i\mu_j}(p_ik_j)^{-1}$; the second is minus the same thing with p_i replaced by $p_i + k_j$. Thus, apart from this simple factor, and, for one term, the overall shift in p_i , the function is just the same as it was before the action of \hat{C}_j . Consequently, the power-counting argument of section 6 goes through essentially unchanged: there is for each classical photon j one extra denominator factor (p_ik_j) coming from the factor $ip_{i\mu_j}(p_ik_j)^{-1}$ just described, but the powers of the various r_i in this denominator factor are exactly cancelled by the numerator factor $(r_1 \dots r_j)$ that we have associated with the vertex \hat{C}_j . Because of this exact cancellation the C -type couplings do not contribute to the power counting. Hence when C -type couplings are allowed the arguments of section 6 lead to the result that the meromorphic part of the function F associated with the quantum photons is of degree at least one in each of the variables r_j . Hence it is infrared convergent.

8. Distortion of the Ω Contours

The proof of infrared finiteness given in sections 6 and 7 depends upon the assumption that the Ω contours can be shifted away from all denominator zeros in the residue factors of any term in the pole-residue decomposition of the Feynman function corresponding to the simple triangle graph, modified by the insertion of an arbitrary number of soft-photon lines, each of which has a quantum coupling on at least one end and a quantum or classical coupling on the other. The proof that such a distortion of the contour is possible requires two generalizations of the available results about the locations of singularities occurring in the terms of the perturbative expansion in field theory.

In the first place, we must deal not only with the Feynman functions themselves, but also with the functions obtained by decomposing, according to the pole-residue theorem, the generalized propagators associated with the three sides of the triangle. For the usual Feynman functions themselves there is available the useful geometric formulation, in terms of Landau diagrams, of necessary conditions for a singularity. In ref. 14 we have developed a generalization of the Landau-diagram condition that covers the more general kinds of functions that arise in our work.

The second needed generalization pertains to the masslessness of photons. If the standard Landau-diagram momentum-space conditions are generalized to include massless particles then the effect of contributions from points where $k_i = 0$, for some i , is to produce a severe weakening of the necessary conditions. But in ref. 14 the needed strong results are obtained in the variables (r_i, Ω_i) introduced in section 6 to prove infrared finiteness.

9. Contributions of the Meromorphic Terms to the Singularity on the Triangle-Diagram Surface $\varphi = 0$.

In this section we describe the contributions to the singularity on the triangle-diagram singularity surface $\varphi = 0$ arising from the meromorphic parts of the three generalized propagators.

The arguments of sections 6, 7, and 8 show that in the typical pole-residue term (5.9) we can distort the contours in the Ω_j variables so as to keep the residue factors analytic, even in the limit when some or all of the r_j 's become zero. In that argument we considered separately an individual half-line, but the argument is ‘local’: it carries over to the full set of six half-lines, with all the $|k_i|$ ordered. Thus for each fixed value of the set of variables $(r_i, \dots, r_n; \Omega_1, \dots, \Omega_n)$ the integration over the remaining variable of integration p gives essentially a triangle-graph function: it gives a function with the same $\log \varphi$ -type singularity that arises from the simple Feynman triangle-graph function itself, with, however, the location of this singularity in the space of the external variables (q_i, q_2, q_3) shifted by an amount (K_1, K_2, K_3) , where the three vectors K_s are related to the photon momenta flowing along the three star lines of the original graph. Specifically, if we re-draw the photon loops so that they pass through *no* star line of the original graph (or equivalently through no star line of the Landau diagram), but pass, instead, *out* of the graph at a vertex v_1, v_2 or v_3 , if necessary, and then define the net momentum flowing out of vertex v_s to be

$$q_s = q_s(k) + K_s, \quad (9.1)$$

where K_s is the net momentum flowing out of vertex v_s along the newly directed photon loops, then, for fixed k , the function in (q_1, q_2, q_3) space will have a normal $\log \varphi$ triangle-diagram singularity along the surface $\varphi(q_1(k), q_2(k), q_3(k)) = 0$. For example, the original singular point at the point \hat{q} in (q_1, q_2, q_3) space will be shifted to the point $(q_1, q_2, q_3) = (\hat{q}_1, \hat{q}_2, \hat{q}_3) + (K_1, K_2, K_3)$. This shift in the external variables q 's shifts the momentum flowing along the three star lines to the values they would have if the photon moments k_i were all zero: it shifts the kinematics back to the one where no photons are present.

It is intuitively clear that the smearing of the location of this $\log \varphi$ singularity caused by the integration of the variables k_i will generally produce a

weakening of the log singularity at $\varphi(q) = 0$. For, in general, only the endpoint $r_1 = 0$ of the r_1 integration will contribute to the singularity at $\varphi(q) = 0$, and there is no divergence at $r_1 = 0$, by power counting, and hence no contribution from this set of measure zero in the domain of integration. The only exception arises from the set of separable graphs. For in these graphs the K_s are all zero, and hence the integrations produce no smearing, and thus no weakening, of the $\log \varphi$ singularity.

To convert this intuitive argument to quantitative form we begin by separating the set of photon lines into two subsets that enter differently into the calculations. Let a *bridge* line in a graph g that corresponds to a term in the pole-residue decomposition (7.1) be a photon line j that ‘bridges’ over a star line: any closed loop in g that contains the photon line segment j , and is completed by charged-particle segments that lie on the triangle G , passes along at least one star line. Let i be the smallest j such that photon line j is a bridge line. (Here we are using the ordering of the full set $(1, 2, \dots, n)$ of photon labels that was specified in (6.1), not the ordering used in (7.1)). Thus each $k_j = \rho_j \Omega_j = r_1 \dots r_j \Omega_j$ that appears in a star-line denominator, and hence in (9.1), contains a factor $\rho_i = r_1 \dots r_i$. Let the set of variables (k_1, \dots, k_{i-1}) be denoted by k_a , and let the set of variables (k_i, \dots, k_n) be denoted by k_b . And let r_a and r_b , and Ω_a and Ω_b be defined analogously. Then the function represented by g can be written in the form

$$F(q) = \prod_{j=i}^n \int_{\Omega_j \tilde{\Omega}_j=1} d\Omega_j \int_0^1 r_j^{e_j} dr_j G(q, \Omega_b, r_b) \quad (9.2)$$

where

$$G(q, \Omega_b, r_b) = \int d^4 p \prod_{j=1}^{i-1} \int_{\Omega_j \tilde{\Omega}_j=1} d\Omega_j \int_0^1 r_j^{e_j} dr_j \prod_{s=1}^3 \frac{1}{p_s(q, \Omega_b, r_b)^2 - m^2 = +i0} R(q, \Omega_b, r_b, \Omega_a, r_a). \quad (9.3)$$

Here R is the product of the three residue factors.

The integrations in (9.2) weaken the logarithmic singularities: it is shown in ref. 15 that the singularity on the surface $\varphi(q) = 0$ is contained in a finite sum of terms of the form $A_m \varphi (\log \varphi)^m$, where m is a positive integer that is no greater than the number of photons in the graph, and A_m is analytic.

10. Operator Formalism.

We have dealt so far mainly with the meromorphic contributions. In order to treat the nonmeromorphic remainder it is convenient to decompose the operator \hat{C}_i into its “meromorphic and “nonmeromorphic” parts, \hat{C}_i^M and \hat{C}_i^N .

The operator \hat{C}_i is defined in (2.5):

$$i\hat{C}_i F(\tilde{p}) = \int_0^1 d\lambda_i \frac{\partial}{\partial p^{\mu_i}} F(p) \quad (10.1)$$

where

$$p = \tilde{p} + \lambda_i k_i. \quad (10.2)$$

Suppose

$$F(p) = A(p)B(p), \quad (10.3)$$

where $A(p)$ is analytic and $B(p)$ is $(p^2 - m^2)^{-1}$. An integration by parts gives

$$\begin{aligned} i\hat{C}_i AB &= \int_0^1 d\lambda_i \left[(\partial_{\mu_i} A) B + A (\partial_{\mu_i} B) \right] \\ &= \int_0^1 d\lambda_i \left[(\partial_{\mu_i} A) - (\partial A / \partial \lambda_i) \int^{\lambda_i} \partial_{\mu_i} \right. \\ &\quad \left. + A (\delta(\lambda_i - 1) - \delta(\lambda_i)) \int^{\lambda_i} \partial_{\mu_i} \right] B, \end{aligned} \quad (10.4)$$

where the difference of delta functions, $(\delta(\lambda_i - 1) - \delta(\lambda_i))$ indicates that one is to take the difference of the integrand at the two end points.

The indefinite integral, computed by the methods used to compute (3.3), (3.6), and (3.7), is

$$\begin{aligned} \int^{\lambda_i} \partial_{\mu_i} B &\equiv \int d\lambda_i \frac{\partial}{\partial p^{\mu_i}} B \\ &= \frac{2p_{\mu_i}}{2pk_i} B - \frac{4(p_{\mu_i} k_i^2 - k_{i\mu_i} p k_i)}{d} \left[\int^{\lambda_i} B + \frac{1}{pk_i} \right]. \end{aligned} \quad (10.5)$$

Because the factor in front of the square bracket in (10.5) is independent of λ_i one can use a second integration by parts (in reverse) to obtain

$$\begin{aligned}
i\hat{C}_i AB = & \int_0^1 d\lambda_i \left[(\partial_{\mu_i} A) - \left(\frac{\partial}{\partial \lambda_i} A \right) \frac{2p_{\mu_i}}{2pk_i} \right. \\
& + A(\delta(\lambda_i - 1) - \delta(\lambda_i)) \frac{2p_{\mu_i}}{2pk_i} \\
& - \frac{4(p_{\mu_i} k_i^2 - k_{i\mu_i} p k_i)}{d} A \\
& \left. + \frac{4(p_{\mu_i} k_i^2 - k_{i\mu_i} p k_i)}{d} \frac{k_i^2 (p^2 - m^2) A}{(pk_i)^2} \right] B, \tag{10.6}
\end{aligned}$$

where the final term comes from the $1/pk_i$ term in the square bracket in (10.5) and has no singularity at $(p^2 - m^2) \equiv B^{-1} = 0$ for $pk_i \neq 0$.

Since all of the λ_i dependence in A is in $p = \tilde{p} + \lambda_i k_i$ we may write

$$\partial A / \partial \lambda_i = (\partial_{\mu_i} A) k^{\mu_i}. \tag{10.7}$$

Hence the first two terms on the right side of (10.6) cancel, and one is left with

$$\hat{C}_i = \hat{C}_i^M + \hat{C}_i^N + \hat{C}_i^R, \tag{10.8}$$

where

$$i\hat{C}_i^M AB = \int_0^1 d\lambda_i \frac{2p_{\mu_i}}{2pk_i} (\delta(\lambda_i - 1) - \delta(\lambda_i)) AB \tag{10.8a}$$

$$i\hat{C}_i^N AB = -\frac{4(p_{\mu_i} k_i^2 - k_{i\mu_i} p k_i)}{d} \int_0^1 d\lambda_i AB \tag{10.8b}$$

$$\begin{aligned}
i\hat{C}_i^R AB &= \frac{4(p_{\mu_i} k_i^2 - k_{i\mu_i} p k_i)}{d} \int_0^1 d\lambda_i \left[\left(\frac{\partial}{\partial \lambda_i} A \right) \right. \\
&\quad \left. - A(\delta(\lambda_i - 1) - \delta(\lambda_i)) \right] \frac{1}{pk_i} \\
&= \frac{4(p_{\mu_i} k_i^2 - k_{i\mu_i} p k_i)}{d} \int_0^1 d\lambda_i \frac{k_i^2 (p^2 - m^2)}{(pk_i)^2} AB. \tag{10.8c}
\end{aligned}$$

Notice that the contribution \hat{C}_i^R cancels the pole at $d = 0$ of the contribution \hat{C}_i^N .

To efficiently manipulate these operators their commutation relations are needed. Recall from section 2 that the operators \widehat{C}_i commute among themselves, as do the \widehat{D}_i :

$$[\widehat{C}_i, \widehat{C}_j] = 0 \quad (10.9a)$$

and

$$[\widehat{D}_i, \widehat{D}_j] = 0. \quad (10.9b)$$

The operators \widehat{C}_i and \widehat{D}_j , properly interpreted, also commute:

$$[\widehat{C}_i, \widehat{D}_j] = 0. \quad (10.9c)$$

To verify (10.9c) note first that \widehat{D}_j acts on generalized propagators (See (2.9)), and, by linearity, on linear superpositions of such propagators. However, Eq. (2.3) shows that the action on such an operand of the operator $(-\partial/\partial p_{\mu_i})$ in \widehat{C}_i is the same as a \widehat{D}_i with $k_i = 0$. Moreover, the replacement $p \rightarrow p + \lambda_i k_i$ commutes with \widehat{D}_j . Thus (10.9c) is confirmed, provided we stipulate that the integrations over the variables λ_i shall be reserved until the end, *after* the actions of all operators \widehat{D}_i and differentiations. In fact, we see from (10.8) that the various partial operators \widehat{C}_i^M , \widehat{C}_j^N , and \widehat{C}_k^R all commute: if we reserve the λ integrations until the end then each of the operations is implemented by multiplying the integrand by a corresponding factor, and those operations commute.

11. Nonmeromorphic Contributions

The D -coupling part of a Q -type coupling is meromorphic. Thus each of the \hat{C} - and \hat{Q} -type couplings can be expressed as by means of (10.8) as sum of its meromorphic, nonmeromorphic, and residual parts. Then the full function can be expanded as a sum of terms in which each coupling is either \hat{C} -type or \hat{Q} -type, and is either meromorphic, nonmeromorphic, or residual. If any factor is residual then the term has no singularity at $(p^2 - m^2) = 0$, and is not pertinent to the question of the singularity structure on $\varphi = 0$. Thus these residual terms can be ignored.

We have considered previously the terms in which every coupling is meromorphic. Here we examine the remainder. Thus terms not having least one nonmeromorphic coupling \hat{C}_i^N or \hat{Q}_i^N are not pertinent: they can also be ignored.

All couplings of the form \hat{Q}_i^M can be shifted to the right of all others, and this product of factors \hat{Q}_i^M can then be re-expressed in terms of the couplings \tilde{Q}_i^M . That is, the terms corresponding to the different orderings of the insertions of the *meromorphic* couplings Q_i^M into the charged-particle lines can be recovered by using (2.9), (2.15), and (5.8). The various couplings \hat{C}_i^N are then represented, apart from the factor standing outside the integral in (10.8b), simply by an integration from zero to one on the associated variable λ_i .

In this paper we are interested in contributions such that every photon has a Q -type coupling on at least one end. In sections 6 and 7 the variables ρ_i 's corresponding to photons i having a \tilde{Q}_i^M -type coupling on (at least) one end were expressed in terms of the variables r_j , and it was shown that the contributions from all of the \tilde{Q}_i^M -type couplings lead to an r_j dependence that is of order at least one in each r_j . The \hat{C}_i^M -type couplings do not upset this result. Thus the general form of the expression that represents any term in the pole-residue expansion of the product of meromorphic couplings \tilde{Q}_i^M and \hat{C}_i^M is

$$\prod_j \int_{\Omega_j \tilde{\Omega}_j=1} d\Omega_j \prod_i \int_0^1 r_i^{e_i} dr_i AB, \quad (11.1)$$

where the e_i are nonnegative integers, and A and B have the forms specified in section 10, provided the Ω contours are distorted in the way described in section 8 and ref. 14. (For convenience, the scale has been defined so that the upper

limit δ of the integration over r_1 is unity.)

For these meromorphic couplings the integrations over the variables λ_i have been eliminated by the factors $\delta(\lambda_i - 1)$ and $\delta(\lambda_i)$. But for any coupling \hat{C}_i^N there will be, in addition to the integration from zero to one on the variable r_i , also an integration from zero to one on the variable λ_i . It comes from (10.8b).

These integrals are computed in ref. 15, and it is shown that the nonmeromorphic contributions lead to the singularities on the triangle diagram singularity surface $\varphi = 0$ that are no stronger than $\varphi(\log \varphi)^{n+1}$, where n is the number of photons in the graph. Even if the log factors from the graphs of different order in e^2 should combine to give a factor like $\varphi^{-(1/137)}$, this factor, when combined with the form $\varphi(\log \varphi)^{n+1}$, would not produce a singularity as strong as the $\log \varphi$ singularity that arises from the separable graphs.

12. Comparison to Other Recent Works

Block and Nordsieck¹² recognized already in 1937 that a large part of the very soft photon contribution to a scattering cross-section was correctly predicted by classical electromagnetic theory. They noted that the process therefore involves arbitrarily large numbers of photons, and that this renders perturbation theory inapplicable. They obtained finite results for the cross section for the scattering of a charged particle by a potential V by taking the absolute-value squared of the matrix element of V between initial and final states in which each charged particle is “clothed” with a cloud of bremsstrahlung soft photons. The two key ideas of Block and Nordsieck are, first, to focus on a physical quantity, such as the observed cross section, with a summation over unobserved very soft photons, and, second, to separate out from the perturbative treatment the correspondence-principle part of the scattering function, which is also the dominant contribution at very low energies.

These ideas have been developed and refined in an enormous number of articles that have appeared during the more than half-century following the paper of Block and Nordsieck. Particularly notable are the works of J. Schwinger¹⁷, Yennie, Frautschi, and Suura¹⁸, and K.T. Mahanthappa¹⁹. Schwinger’s work was the first modern treatment of the infrared divergence problem, and he conjectured exponentiation. Yennie, Frautschi, and Suura, formulated the problem in terms of Feynman’s diagrammatic method, and analyzed particular contributions in detail. They gave a long argument suggesting that their method should work in all orders, but their argument was admittedly nonrigorous, and did not lend itself to easy rigorization. The main difficulties had to do with the failure of their arguments at points where the basic scattering function was singular. These points are precisely the focus of the present work, and our way of separating out the dominant parts leads to remainder terms that are compactly representable, and hence amenable to rigorous treatment. Mahanthappa considered, as do we, closed time loops, and split the photons into hard and soft photons, and constructed an electron Green’s function in closed form for the soft-photon part to do perturbation theory in terms of the hard part.

General ideas from these earlier works are incorporated into the present work. But our logical point of departure is the article of Chung²⁰ and of Kibble⁴.

Chung was the first to treat the scattering amplitudes directly, instead of transition probabilities, and to introduce, for this purpose, the coherent states of the electromagnetic field. Kibble first exhibited the apparent break-down of the pole-factorization property in QED. The present work shows that this effect is spurious: the non-pole form does not arise, at least in the case that we have examined in detail, if one separates off for nonperturbative treatment not the approximate representation of the correspondence-principle part used by Chung and Kibble, but rather an accurate expression that is valid also in case the scattering process is macroscopic, and that therefore involves no replacement of factors $\exp ikx$ by anything else.

The works mentioned above are not directly comparable to present one because they do not address the question at issue here, which is the large-distance behaviour of quantum electrodynamics, and in particular the dominance at large distances of a part that conforms to the correspondence principle and enjoys the pole-factorization property. The validity of these principles in quantum electrodynamics is essential to the logical structure of quantum theory: the relationship between theory and experiment would become ill-defined if these principles were to fail. These principles are important also at the practical level. The domain of physics lying between the atomic and classical regimes is becoming increasingly important in technology. We therefore need to formulate the computational procedures of quantum electrodynamics in a way that allows reliable predictions to be made in this domain. Moreover, the related “problem of measurement” is attracting increasing attention among theorists and experimentalists. The subject of this work is precisely the subtle mathematical properties of this quantum-classical interface in the physical theory that actually controls it. Finally, the problem of the effects of massless particles in gauge theories is an issue of mounting theoretical importance. Theorists need to have an adequate treatment of this mathematically delicate problem in our premier physical theory, quantum electrodynamics, which serves as a model for all others.

Kulish and Faddeev²¹ have obtained a finite form of quantum electrodynamics by modifying the dynamics of the asymptotic states. For our purposes it is not sufficient merely to make the theory finite. We are interested in the nature of singularities, and the related question of the rates of fall-off for large spacetime separations. To obtain a sufficiently well-controlled computational procedure,

in which no terms with spurious rates of fall off are introduced by an unphysical separation of the problem into parts, it was important, in our definition of the classical part, to place the sources of the classical radiation field, and of the classical “velocity” fields, at their correct locations. The needed information about the locations of the scattering sites is not naturally contained in the asymptotic states: the scattering events can involve both “in” and “out” particles together, and perhaps also internal particles as well. We bring in the correct locations of the scattering sites by rearranging the terms of the coordinated-space perturbative expansion of the full scattering operator itself, rather than by redefining the initial and final states of the S-matrix.

d’Emilio and Mintchev²² have initiated an approach that is connected to the one pursued here. They have considered charged-field operators that are nonlocal in that each one has an extra phase factor that is generated by an infinite line integral along a ray that starts at the field point x . Their formula applied to the case of a product of three current operators located at the three vertices (x_1, x_2, x_3) of our closed triangular loop *could* be made to yield precisely the phase that appears in Eq. (1.7) of ref. 11. However, that would involve making the direction of the ray associated with each field operator $\psi(x)$ depend upon the argument of the *other* field operator in the coordinate-space Green’s function $\langle T\psi(x)\bar{\psi}(x') \rangle_0$ in which it appears.

d’Emilio and Mintchev do not follow that tack. Instead, they keep the direction of the ray associated with each field operator $\psi(x)$ fixed, *then* go to momentum space, and then find, for some simple cases (charged-particle propagation and vertex correction), that reasonable results are obtained only if the directions of the rays associated with the charged-particle operators that create or destroy the electrons are set equal to the momenta of the particles that are created or destroyed.

Of course, charged-particle propagators generally occur under integral signs, whereas the directions of the rays are treated as constants. If these “constant” directions are allowed to depend upon the momentum p then the inverse Fourier transform would, of course, not yield the original coordinate-space Feynman function.

An “intuitive” reason was given why the one particular choice of the direc-

tions of the rays gives reasonable answers. It relies on the idea of “the classical currents responsible for the emission of soft photons”. But classical-current arguments ought to be formulated in coordinate space.

Such a formulation (i.e., a coordinate-space formulation) would suggest letting the direction of the ray that occurs in the d’Emilio-Mintchev formula be the direction of the line between the two arguments x and x' of the coordinate-space charged-particle propagator. Then, due to a partial cancellation, the two infinite line integrals would collapse to a single finite line integral running between the two points x and x' . Then, in the case of our triangular closed loop, the phase factors associated with the lines on the three sides of the triangle would combine to give just the phase factors appearing in (1.7) of reference 11.

This coordinate-space procedure, which would seem to be the physically reasonable way to proceed, would bring the d’Emilio-Mintchev formulation to the first stage of the work pursued here and in reference 11.

The problem of formulating quantum electrodynamics in an axiomatic field-theoretic framework has been examined by Fröhlich, Morchio, and Strocchi⁸ and by D. Buchholz⁹, with special attention to the non-local aspects arising from Gauss’ law. Their main conclusion, as it relates to the present work, is that the energy-momentum spectrum of the full system can be separated into two parts, the first being the photonic asymptotic free-field part, the second being a remainder that: (1), is tied to charged particles; (2), is nonlocal relative to the photonic part; and (3), can have a discrete part corresponding to the electron/positron mass. This separation is concordant with the structure of the QED Hamiltonian, which has a photonic free-field part and an electron/positron part that incorporates the interaction term $eA^\mu J_\mu$, but no added term corresponding to the non-free part of the electromagnetic field. It is also in line with the separation of the classical electromagnetic field, as derived from the Liénard-Wiechert potentials, into a “velocity” part that is attached (along the light cone) to the moving source particle, and an “acceleration” part that is radiated away. It is the “velocity” part, which is tied to the source particle, and which falls off only as r^{-1} , that is the origin of the “nonlocal” infraparticle structure that introduces peculiar features into quantum electrodynamics, as compared to simple local field theories.

In the present approach, the quantum analog of this entire classical structure is incorporated into the formula for the scattering operator by the unitary factor $U(L)$. It was shown in ref. 11, Appendix C, that the non-free “velocity” part of the electromagnetic field generated by $U(L)$ contributes in the correct way to the mass of the electrons and positrons. It gives also the “Coulomb” or “velocity” part of the interaction between different charged particles, which is the part of the electromagnetic field that gives the main part of Gauss’ law asymptotically. Thus our formulas supply in a computationally clean way these “velocity field” contributions that seem so strange when viewed from other points of view.

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